Relation Algebras from Cylindric Algebras, II

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Abstract

We prove, for each $4 \leq n < \omega$, that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ cannot be defined, using only finitely many first-order axioms, relative to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. The construction also shows that for $5 \leq n < \omega$, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is not finitely axiomatisable over RA_n, and that for $3 \leq m < n < \omega$, $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$. In consequence, for a certain standard n-variable first-order proof system $\vdash_{m,n}$ of m-variable formulas, there is no finite set of m-variable schemata whose m-variable instances, when added to $\vdash_{m,n}$ as axioms, yield $\vdash_{m,n+1}$.

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1 Introduction and summary

In the nineteenth century there were two main approaches to the formalization of quantification in logic. The first approach, due to de Morgan and taken up by Peirce, led to what we now call relation algebra (see [18] for an account of the early history of relation algebra); the other approach, due to Frege, became the standard formalism of first-order logic with its explicit universal and existential quantifiers. Both can express quantification, though in different ways — in the algebraic approach to binary relations we use the composition of binary relations. For example, in first-order logic we can say 'there exists a person who is my parent and your sibling', which could be expressed in relation algebra as 'you are either my uncle or my aunt'.

Then, in the twentieth century, first-order logic was given an algebraic setting in the framework of *cylindric algebra* [6, 7]. So we now have two main algebraic formalisms for relations of various ranks: relation algebras constitute an algebraization of binary relations and n-dimensional cylindric algebras are an algebraization of n-ary relations. Ever since these classes of algebras were defined, researchers have investigated the connections between them [17, for example]. The *relation algebra reduct* is a known way of turning a cylindric algebra into a relation algebra: we extract the essentially binary relations of the cylindric algebra and interpret the relation algebra operations on them by suitable cylindric algebra terms. But the question arises as to when a given relation algebra can be obtained as such a reduct — or at least as a subalgebra of such a reduct. To put it another way, if $\mathcal{C}$ is an n-dimensional cylindric

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algebra and $\mathcal{A}$ is a subalgebra of the relation algebra reduct $\mathfrak{Ra}(\mathcal{C})$ of $\mathcal{C}$, is there a trace purely within $\mathcal{A}$ of its origin as a n-dimensional cylindric algebra? In symbols: for a relation algebra $\mathcal{A}$, the question is to tell whether or not $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, the class of subalgebras of relation algebra reducts of n-dimensional cylindric algebras. Certainly it is not easy to say, even for a finite relation algebra $\mathcal{A}$, whether $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ $(n \geq 5)$. The proof in [9] that, for finite relation algebras, the representability problem is undecidable can be extended to show also that membership of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is undecidable (for finite relation algebras, for $n \geq 5$).[1]

A theoretical tool to determine membership of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ can nevertheless be obtained. In [16], Maddux defined an *n-dimensional relational basis* of an atomic relation algebra to be a set of *n-dimensional basic matrices* or *atomic n-networks* (complete directed graphs on n nodes with edges labelled by atoms of the algebra and satisfying some given consistency conditions) with certain closure properties. RA_n denotes the class of subalgebras of atomic relation algebras with such a basis of atomic networks. However, although $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$, it turns out that there are relation algebras in RA_n which do not belong to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ (see remark 25 below). Maddux also defined an *n-dimensional cylindric basis* [14, 20] to be an n-dimensional relational basis satisfying an additional 'amalgamation' closure condition which allows us to 'glue together' two atomic networks from the cylindric basis along isomorphic subnetworks, so long as the resulting atomic network is restricted to just n nodes. Maddux showed that the atomic networks in an n-dimensional cylindric basis could be considered as the atoms of an n-dimensional cylindric algebra, with natural cylindric algebra operations on them. The amalgamation condition ensures that the cylindrifiers in this induced cylindric algebra commute with each other ($\mathsf{c}_i\mathsf{c}_j x = \mathsf{c}_j\mathsf{c}_i x$). Any subalgebra of a relation algebra with an n-dimensional cylindric basis belongs to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, but we conjecture that the converse fails.

To obtain a characterisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ in Part I of this paper [8], we modified the definitions above and defined n-dimensional hypernetworks and hyperbases. An *n-dimensional hypernetwork* is similar to an atomic n-network, but it also has 'hyperlabels' on longer sequences of nodes, allowing us to express higher-order constraints on amalgamation. An *n-dimensional hyperbasis* is now defined in just the same way as an n-dimensional cylindric basis, the only difference being that it consists of n-dimensional hypernetworks instead of atomic n-networks. As with cylindric bases, it is easy to see that the hypernetworks in a hyperbasis form the atom structure of a cylindric algebra. But the additional hyperlabels have the effect of making the amalgamation condition weaker. There may be two hypernetworks in a hyperbasis with subnetworks whose restrictions to binary edge labels are isomorphic, but which are not isomorphic as hypernetworks. If we threw away the hyperlabels and considered the basis as a cylindric basis then an amalgum would be demanded; but as a hyperbasis, an amalgum is not required. This weakening of the effect of the amalgamation condition helped us prove the converse: the canonical extension of any relation algebra reduct of an n-dimensional cylindric algebra has an n-dimensional hyperbasis. Thus, we showed that for any relation algebra $\mathcal{A}$, $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if its canonical extension $\mathcal{A}^+$ has an n-dimensional hyperbasis, if and only if some atomic extension of $\mathcal{A}$ has an n-dimensional hyperbasis. If $\mathcal{A}$ is finite then $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if $\mathcal{A}$ has an n-dimensional hyperbasis (though the basis need not be finite). So the additional hyperlabelling that is present in a hypernetwork can be used to

[1] Interestingly, the situation is quite different with RA_n, the class of subalgebras of relation algebras with relational bases. An algorithm given by Maddux shows that it is decidable whether a finite algebra belongs to this class or not.

obtain a characterisation of the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.

We now have two sequences of classes (actually canonical varieties [16, 8]) approximating the class RRA of all *representable relation algebras:*

$$\begin{aligned} \mathsf{RA} &= \mathsf{RA}_4 \supseteq \mathsf{RA}_5 \supseteq \mathsf{RA}_6 \supseteq \cdots && (1) \\ \mathsf{RA} &= \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_6 \supseteq \cdots && (2) \end{aligned}$$

and $\mathsf{RRA} = \bigcap_{4\leq n<\omega} \mathsf{RRA} = \bigcap_{4\leq n<\omega} \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. All of these inclusions are known to be strict [19, 11]. Also, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$ for each n [8, proposition 61], and the inclusion is strict for $n \geq 5$, as we remarked above.

In [10], we established that RA_{n+1} is not finitely axiomatisable over RA_n, for each $n \geq 4$. Here, we will prove that the class $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ cannot be defined using only a finite number of axioms relative to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, for $n \geq 4$. In other words, there is no first-order sentence σ_n such that for all $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ we have $\mathcal{A} \models \sigma_n \Leftrightarrow \mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. See theorem 27 below.

The key point in our argument is that just as the existence of a hyperbasis of higher and higher dimension approximates representability of a relation algebra more and more closely, so, for a fixed dimension, the existence of a hyperbasis of that dimension can itself be approximated. We do this by games. We devise a two-player game that constructs an approximation to an n-dimensional hyperbasis for an atomic relation algebra $\mathcal{A}$ and tests its closure properties (amalgamation, etc) r times. For countable $\mathcal{A}$, if it survives ω tests it is a genuine hyperbasis.

Then, for each $n \geq 4$, we will construct finite relation algebras $\mathfrak{A}(n,r)$ $(1 \leq r < \omega)$ which have an approximation to an $(n+1)$-dimensional hyperbasis that survives r tests of this kind, but which have no genuine hyperbasis. So the $\mathfrak{A}(n,r)$ are never actually in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ (for finite algebras, this is guaranteed by the lack of a hyperbasis), but as r increases, they get closer and closer to it. It follows that an ultraproduct of them *is* in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$, so by Łoś' theorem, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ is not finitely axiomatisable. We will even show that $\mathfrak{A}(n,r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \cap \mathsf{RA}_{n+1}$ for all r, so that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, RA_{n+1}, and indeed any class containing $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \cap \mathsf{RA}_{n+1}$.

Much the same story can be told for cylindric algebras. Monk showed in [22] that for finite $m \geq 3$, the class RCA_m of representable m-dimensional cylindric algebras is not finitely axiomatisable. But by insisting that an m-dimensional cylindric algebra is a subalgebra of the 'neat reduct' of an n-dimensional cylindric algebra, for larger and larger finite n, we get closer and closer to RCA_m. In symbols, we have

$$\mathsf{CA}_m = \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_m \supseteq \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{m+1} \supseteq \cdots \qquad (3)$$

and $\bigcap_{3\leq m\leq n<\omega} \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n = \mathsf{RCA}_m$. Analogous questions to those for $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ now arise about whether the inclusions in (3) are proper and non-finitely axiomatisable. In [2], Andréka proved that $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \neq \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ if $3 \leq m \leq n < 2m < \omega$. In [3, theorem 2], she proved that for $m \geq 3$, $n \geq m+2$, $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$ is not finitely axiomatisable and cannot be axiomatised at all with a set of prenex universal sentences using only a bounded number of variables (a similar result for relation algebras was proved in [13]). Ahmed proves in [1] that $\mathfrak{Nr}_m\mathsf{CA}_n$ is not even elementary, for $1 < m < n$. In [11] it was shown that for any finite $m \geq 3$, all the inclusions in (3) are proper; this is a corollary of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_3 \supset \mathbf{S}\mathfrak{Ra}\mathsf{CA}_4 \supset \mathbf{S}\mathfrak{Ra}\mathsf{CA}_5 \supset \cdots$. In theorem 27 below, we show that all inclusions in (3) but the first are non-finitely axiomatisable:

for $3 \leq m < n < \omega$, there is no first-order sentence $\rho_{m,n}$ such that for all $\mathcal{C} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$, we have $\mathcal{C} \models \rho_{m,n} \iff \mathcal{C} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$. (Interestingly, the inequality $m < n$ is indispensible here: Andréka shows that $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{m+1}$ *is* finitely axiomatisable [2]. We use the restriction $m < n$ only once in our proof.)

Our proof is again by games. The game mentioned above is designed to test a rather stronger property than the simple existence of a hyperbasis: this is helpful in obtaining the non-finite axiomatisability result about neat reducts. For $3 \leq m \leq n < \omega$, $0 < r < \omega$, we can define an m-dimensional cylindric algebra $\mathfrak{C}_r$ by taking as atom structure the m-dimensional hyperbasis of all m-dimensional hypernetworks over $\mathfrak{A}(n,r)$. Much as for $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, we show that each $\mathfrak{C}_r \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ but for $m < n$ we show that a non-principal ultraproduct of the $\mathfrak{C}_r$ is in $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$. It follows, for $3 \leq m < n < \omega$, that $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$.

This result has an application in finite variable proof theory. In [7, 5], a certain natural Hilbert system $\vdash_{m,n}$ is given, to prove m-variable formulas in an m-ary relational signature. Proofs in $\vdash_{m,n}$ can use up to n variables. Any m-variable formula φ corresponds in a natural way to a CA_m-term $\widehat{\varphi}$, and it has been shown that $\vdash_{m,n} \varphi$ if and only if $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \models \widehat{\varphi} = 1$. Because $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1} \subset \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$, it follows that $\vdash_{m,n+1}$ is strictly stronger than $\vdash_{m,n}$. Because for $3 \leq m < n < \omega$, $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$, it follows that there is no finite set Σ of m-variable schemata such that the system '$\Sigma \vdash_{m,n}$', with all m-variable instances of schemata in Σ as additional axioms of $\vdash_{m,n}$, proves the same theorems as $\vdash_{m,n+1}$.

As we said, this paper continues the work of [10, 11] and Part I of this paper [8], but *we intend that it can be read independently of them.* We will borrow the hyperbasis characterisation of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ and a few other minor results from Part I, and we will assume familiarity with relation algebras and such, but otherwise the paper is self-contained.

Outline of paper In section 2, we recall various definitions: for example, of the m-dimensional neat reduct $\mathfrak{Nr}_m\mathcal{C}$ and the relation algebra reduct $\mathfrak{Ra}\mathcal{C}$ of a cylindric algebra $\mathcal{C}$, and of a hypernetwork and hyperbasis. We quote a result from [8]: a relation algebra $\mathcal{A}$ belongs to $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ if and only if it embeds in some relation algebra (which can be taken to be its canonical embedding algebra $\mathcal{A}^+$) that has an n-dimensional hyperbasis. We prove results relating hyperbases to neat reducts. Then, in section 3 we construct the finite relation algebras $\mathfrak{A}(n,r)$ for $4 \leq n < \omega$ and $r < \omega$, and the m-dimensional cylindric algebras $\mathfrak{C}_r$ for $3 \leq m < n$, and show that:

1. $\mathfrak{A}(n,r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ and $\mathfrak{C}_r \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$ (theorem 15 and corollary 17). This is done by exhibiting an n-dimensional hyperbasis for $\mathfrak{A}(n,r)$.

2. For $r > 0$, $\mathfrak{A}(n,r) \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ and $\mathfrak{C}_r \notin \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ (theorem 18 and corollary 20). This is proved by obtaining a contradiction from the assumption that $\mathfrak{A}(n,r)$ has an $(n+1)$-dimensional hyperbasis; this suffices because $\mathfrak{A}(n,r)$ is finite and so isomorphic to $\mathfrak{A}(n,r)^+$. The result for $\mathfrak{C}_r$ follows.

In section 4, we define the r-round game $G_r^{m,n}(\mathcal{A},\Lambda)$, for an atomic relation algebra $\mathcal{A}$ and non-empty set Λ of labels for our hypernetworks, and argue, for countable $\mathcal{A}$ and Λ, that a winning strategy for the second player (called '$\exists$') in $G_\omega^{m,n}(\mathcal{A},\Lambda)$ is equivalent to the existence of an n-dimensional hyperbasis $\mathcal{H}$ such that

($\mathcal{E}$) any m-dimensional hypernetwork extends to an n-dimensional hypernetwork in $\mathcal{H}$.

We then prove:

3. $\exists$ has a winning strategy in $G_r^{m,n+1}(\mathfrak{A}(n,r),\Lambda)$, for all $3 \leq m < n < \omega$ and $r < \omega$ (theorem 24).

In section 5 we prove our non-finite axiomatisability result from these three facts about $\mathfrak{A}(n,r)$. Using (3), ultraproducts, and elementary chains, we obtain a countable relation algebra $\mathcal{B}$ elementarily equivalent to $\prod_{r<\omega}\mathfrak{A}(n,r)/D$, such that $\exists$ has a winning strategy in $G_\omega^{m,n+1}(\mathcal{B},\Lambda)$; here, D is any non-principal ultrafilter over ω. Thus, $\mathcal{B}$ has an $(n+1)$-dimensional hyperbasis, so $\mathcal{B} \subseteq \mathfrak{Ra}\mathcal{D}$ for some $\mathcal{D} \in \mathsf{CA}_{n+1}$ and $\mathcal{B} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. At the same time, using $(\mathcal{E})$, we obtain countable $\mathcal{C} \subseteq \mathfrak{Nr}_m\mathcal{D}$ elementarily equivalent to $\prod_{r<\omega}\mathfrak{C}_r/D$. Thus, $\mathcal{C} \in \mathfrak{Nr}_m\mathsf{CA}_{n+1}$. Using (1,2), the non-finite axiomatisability results then follow by Łoś' theorem.

Section 6 applies the result on neat reducts to finite variable proof theory.

Notation CA_n denotes the class of all n-dimensional cylindric algebras. For $\mathcal{C} \in \mathsf{CA}_n$, $\mathfrak{Ra}\mathcal{C}$ denotes the relation algebra reduct of $\mathcal{C}$ and for $m \leq n$, $\mathfrak{Nr}_m\mathcal{C}$ denotes the neat reduct to m dimensions of $\mathcal{C}$, defined in definition 1. If $X \subseteq \mathsf{CA}_n$ then $\mathfrak{Ra}X$, $\mathfrak{Nr}_mX$ denote the classes $\{\mathfrak{Ra}\mathcal{C} : \mathcal{C} \in X\}$, $\{\mathfrak{Nr}_m\mathcal{C} : \mathcal{C} \in X\}$ respectively. If X is any class of algebras then $\mathbf{S}X$ denotes the class of all isomorphic copies of subalgebras of members of X. If $\mathcal{A}$ is any boolean algebra with operators, $\mathcal{A}^+$ denotes the canonical embedding algebra of $\mathcal{A}$. $At\mathcal{A}$ denotes the set of atoms of (the boolean part of) $\mathcal{A}$. We generally identify (notationally) an algebra with its domain; but if $\mathcal{A}$ and $\mathcal{B}$ are algebras, we still write $\mathcal{A} \subseteq \mathcal{B}$ to denote that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, or occasionally that $\mathcal{A}$ is isomorphic to such a subalgebra. Most of the other notation we use is in conformity with that of [6, 7].

Ordinals in this paper are finite or sometimes ω; an ordinal is the set of smaller ordinals. For any set X and ordinal n, ${}^nX$ denotes the set of functions $: n \to X$, which we view as the set of n-tuples of elements of X. ${}^{<n}X$ denotes $\bigcup_{m<n}{}^mX$ and ${}^{\leq n}X$ denotes ${}^{<n+1}X$. If $\bar{x}$ is an n-tuple, we will write $|\bar{x}| = n$, and we always assume implicitly that $\bar{x} = (x_0,\ldots,x_{n-1})$. We write $rg(\bar{x})$ for the set $\{x_0,\ldots,x_{n-1}\}$. For any function $f : X \to Y$, $n < \omega$, and n-tuple $\bar{x} = (x_0,\ldots,x_{n-1}) \in {}^nX$, we write $f(\bar{x})$ for the n-tuple $(f(x_0),\ldots,f(x_{n-1}))$. We also write $rg(f)$ for the range of f, and $f\restriction_{X'}$ for the restriction of f to a subset X' of X.

We will generally use n for the dimension; for particular indices $< n$ or points of hypernetworks, we usually use x, y, z and occasionally i, j, k. For $i, j < n$, $[i/j]$ is the map $[i/j] : n \to n$ (so $[i/j]$ depends implicitly on n) defined by $i \mapsto j$, and $k \mapsto k$ for $k < n$, $k \neq i$.

2 Relation Algebras and Cylindric Algebras

We assume a basic knowledge of these algebras — see [17] for an introduction, and [6, 7] for a comprehensive study of cylindric algebras. Here, we give some basic facts about them, and the hyperbases that we use to link them together. Fix, for this section, a finite ordinal $n \geq 2$; occasionally we will require $n \geq 4$.

2.1 Reducts of cylindric algebras to lower dimensions

Given an n-dimensional cylindric algebra $\mathcal{C}$, and some $m \leq n$, it is possible to construct an m-dimensional cylindric algebra $\mathfrak{Nr}_m\mathcal{C}$ from $\mathcal{C}$. For $n \geq 4$ we can also construct a relation algebra $\mathfrak{Ra}\mathcal{C}$ from $\mathcal{C}$. In the following, we use the *j-for-i substitutor* s^i_j for $i, j < n$, defined by

$$\mathsf{s}^i_j x = \begin{cases} x, & \text{if } i = j \\ \mathsf{c}_i(x \cdot \mathsf{d}_{ij}), & \text{otherwise.} \end{cases}$$

DEFINITION 1 *Let $\mathcal{C}$ be any n-dimensional cylindric algebra.*

- *For $m \leq n$, we define the* neat reduct of $\mathcal{C}$ to m dimensions, $\mathfrak{Nr}_m\mathcal{C}$, *to be the* CA_m-*type algebra with domain $Nr_m\mathcal{C} = \{c \in \mathcal{C} : \mathsf{c}_i c = c \text{ for all } m \leq i < n\}$ and operators $+, -, 0, 1, \mathsf{d}_{ij}, \mathsf{c}_i$ $(i, j < m)$ inherited from $\mathcal{C}$. In particular, for $i, j < m$, the diagonal is defined by $\mathsf{d}^{\mathfrak{Nr}_m\mathcal{C}}_{ij} = \mathsf{d}^{\mathcal{C}}_{ij}$ $(\in Nr_m\mathcal{C})$, and the cylindrifiers are defined by $\mathsf{c}^{\mathfrak{Nr}_m\mathcal{C}}_i c = \mathsf{c}^{\mathcal{C}}_i c$, for $c \in Nr_m\mathcal{C}$ and $i < m$. $\mathfrak{Nr}_m\mathcal{C}$ is an m-dimensional cylindric algebra [6, definition 2.6.28 and theorem 2.6.27].*

- $\mathsf{S}\mathfrak{Nr}_m\mathsf{CA}_n$ *denotes the class of all m-dimensional cylindric algebras $\mathcal{C}$ such that there is $\mathcal{D} \in \mathsf{CA}_n$ with $\mathcal{C} \subseteq \mathfrak{Nr}_m\mathcal{D}$.*

- *When $n \geq 3$, the* relation algebra reduct $\mathfrak{Ra}\mathcal{C}$ *is defined to be the algebra $\langle Nr_2\mathcal{C}, 0, 1, +, -, 1', \smile, ; \rangle$, where*

  - $1' = \mathsf{d}_{01}$ $(\in Nr_2\mathcal{C})$
  - *converse is defined by $\breve{a} = \mathsf{s}^2_0\mathsf{s}^0_1\mathsf{s}^1_2 a$, for $a \in Nr_2\mathcal{C}$.*
  - *composition is defined by $a_1 ; a_2 = \mathsf{c}_2(\mathsf{s}^1_2 a_1 \cdot \mathsf{s}^0_2 a_2)$, for $a_1, a_2 \in Nr_2\mathcal{C}$.*

  For $n \geq 4$, the algebra $\mathfrak{Ra}(\mathcal{C})$ can be checked to be a relation algebra [7, 5.3.8].

- $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ *denotes the class of all relation algebras $\mathcal{A}$ such that there is $\mathcal{C} \in \mathsf{CA}_n$ with $\mathcal{A} \subseteq \mathfrak{Ra}(\mathcal{C})$.*

2.2 Hypernetworks

Until section 3, $\mathcal{A}$ will be an atomic relation algebra.

DEFINITION 2

- *Let Λ be a non-empty set disjoint from $At\mathcal{A}$. An n-*dimensional Λ-hypernetwork N over *$\mathcal{A}$ is a map $N : {}^{\leq n}n \to At\mathcal{A} \cup \Lambda$ such that $N(\bar{x}) \in At\mathcal{A}$ if and only if $|\bar{x}| = 2$, for any $\bar{x} \in {}^{\leq n}n$, and with the following properties, for all $x, y, z < n$:*

  1. *$N(x, x) \leq 1'$ (or more strictly, $N(\bar{x}) \leq 1'$, where $\bar{x} = (x, x) \in {}^2n$)*
  2. *$N(x, y) \leq N(x, z) ; N(z, y)$*

  and

  3. *If $\bar{x}, \bar{y} \in {}^{\leq n}n$, $|\bar{x}| = |\bar{y}| \neq 2$, and $N(x_i, y_i) \leq 1'$ for all $i < |\bar{x}|$ (which we write as $\bar{x} \sim_N \bar{y}$), then $N(\bar{x}) = N(\bar{y})$.*

- *For $n < \omega$, any atomic relation algebra $\mathcal{A}$, and any non-empty set Λ disjoint from $At\mathcal{A}$, we write $H_n(\mathcal{A}, \Lambda)$ for the set of all n-dimensional Λ-hypernetworks over $\mathcal{A}$. When Λ is given in the context we may simply write $H_n(\mathcal{A})$ for short.*

When the context is clear, we will drop $\Lambda, \mathcal{A}, n$ and simply say that N is a hypernetwork. Whenever we mention Λ-hypernetworks (or, later, Λ-hyperbases) over $\mathcal{A}$, we will assume that Λ is disjoint from $At\mathcal{A}$. In context, x, y, z are called *nodes* of the hypernetwork N; a sequence $\bar{x} \in {}^{\leq n}n$ with $|\bar{x}| = 2$ is called an *edge*, and if $|\bar{x}| \neq 2$ it is called a *hyperedge*. Labels of hyperedges may be called *hyperlabels*. So we are thinking of an n-dimensional hypernetwork as a labelled hypergraph with set of nodes n.

In the definition of a hypernetwork, above, the first two conditions force the edge labelling to form an n-dimensional *basic matrix* or atomic network [14, 20]. They ensure that the third condition also holds when $|\bar{x}| = |\bar{y}| = 2$.

The following lemma is a straightforward exercise and was proved in [8, lemma 14].

LEMMA 3 *Let N be an n-dimensional hypernetwork and let $x, y < n$. Then $N(x, y) = N(y, x)^\smile$.*

2.3 Hyperbases

DEFINITION 4 *Let M, N be n-dimensional hypernetworks. For $x, y < n$, we write $N \equiv_x M$ if $N(\bar{y}) = M(\bar{y})$ for all $\bar{y} \in {}^{\leq n}(n \setminus \{x\})$, and and $N \equiv_{xy} M$ if $N(\bar{y}) = M(\bar{y})$ for all $\bar{y} \in {}^{\leq n}(n \setminus \{x, y\})$. More generally, if $\bar{x} \in {}^{<\omega}n$ is any sequence we write $N \equiv_{\bar{x}} M$ if $N(\bar{y}) = M(\bar{y})$ for all $\bar{y} \in {}^{\leq n}(n \setminus rg(\bar{x}))$.*

In Maddux's terminology, $M \equiv_x N$ means that M and N 'agree off of x'.

DEFINITION 5 *Let Λ be a non-empty set disjoint from $At\mathcal{A}$. An* n-dimensional Λ-hyperbasis *for $\mathcal{A}$ is a set $\mathcal{H}$ of n-dimensional Λ-hypernetworks over $\mathcal{A}$, such that*

1. *for all $a \in At(\mathcal{A})$ there is $N \in \mathcal{H}$ such that $N(0, 1) = a$*
2. *for all $N \in \mathcal{H}$, for all $x, y, z < n$ with $z \neq x, y$, and for all $a, b \in At(\mathcal{A})$ such that $N(x, y) \leq a ; b$, there is $M \in \mathcal{H}$ with $M \equiv_z N$ and $M(x, z) = a$, $M(z, y) = b$*
3. *for all $M, N \in \mathcal{H}$ and $x, y < n$ with $M \equiv_{xy} N$, there is $L \in \mathcal{H}$ such that $M \equiv_x L \equiv_y N$.*

We say that $\mathcal{A}$ has an n-dimensional hyperbasis *if there exists an n-dimensional Λ-hyperbasis for $\mathcal{A}$, for some set Λ as above.*

This is equivalent to [8, definition 15]. We now quote part of the main results of Part I of this paper [8, theorem 1, remark 55].

THEOREM 6 *Let $\mathcal{A}$ be any relation algebra, and $n \geq 4$. The following are equivalent:*

1. *$\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$*
2. *$\mathcal{A}$ embeds in some atomic relation algebra $\mathcal{B}$ that has an n-dimensional hyperbasis.*
3. *The canonical embedding algebra $\mathcal{A}^+$ of $\mathcal{A}$ has an n-dimensional hyperbasis.*

(1) $\Rightarrow$ (3) in the theorem is proved by technical calculations in cylindric algebras, and (3) $\Rightarrow$ (2) is trivial. But (2) $\Rightarrow$ (1) can easily be proved by making the hypernetworks of a hyperbasis into an atom structure for a cylindric algebra, much as in [20]. As we will need this construction here, we give brief details of it.

Definition 7 *Let $\mathcal{H}$ be a set of n-dimensional Λ-hypernetworks over an atomic relation algebra $\mathcal{A}$. We define an algebra $\mathfrak{Ca}(\mathcal{H})$ of the type of n-dimensional cylindric algebras. The domain of $\mathfrak{Ca}(\mathcal{H})$ is the power set $\wp(\mathcal{H})$. The boolean operations are defined as expected (as complement and union of sets). For $i, j < n$, we define the diagonal*

$$\mathsf{d}_{ij} = \{N \in \mathcal{H} : N(i,j) \leq 1'\},$$

and for $i < n$ we define the cylindrifier c_i by

$$\mathsf{c}_i S = \{N \in \mathcal{H} : \exists M \in S(N \equiv_i M)\}, \quad \text{for } S \subseteq \mathcal{H}.$$

Proposition 8 *If $\mathcal{H}$ is an n-dimensional hyperbasis for $\mathcal{A}$ then $\mathfrak{Ca}(\mathcal{H})$ is an n-dimensional cylindric algebra, and $\mathcal{A}$ embeds into $\mathfrak{Ra}\mathfrak{Ca}(\mathcal{H})$ via $a \mapsto \{N \in \mathcal{H} : N(0,1) \leq a\}$.*

The proof of this proposition follows the proof of [20, theorem 10] closely and we omit it.

2.4 Hyperbases and neat reducts

We now examine neat reducts of algebras of the form $\mathfrak{Ca}(\mathcal{H})$. Fix $3 \leq m \leq n$.

Definition 9

1. If N is an n-dimensional Λ-hypernetwork over $\mathcal{A}$, then $N{\restriction_m}$ denotes the restriction of the map N to ${}^{\leq m}m$. Clearly, N_m is an m-dimensional Λ-hypernetwork.
2. If $\mathcal{H}$ is an n-dimensional hyperbasis for $\mathcal{A}$, we write $\mathcal{H}{\restriction_m}$ for $\{N{\restriction_m} : N \in \mathcal{H}\}$.

Our aim is to prove that $\mathcal{H}{\restriction_m}$ is an m-dimensional hyperbasis for $\mathcal{A}$ and that $\mathfrak{Ca}(\mathcal{H}{\restriction_m}) \subseteq \mathfrak{Nr}_m\mathfrak{Ca}(\mathcal{H})$ (lemma 12 and proposition 14 below).

Definition 10 *If N is an n-dimensional Λ-hypernetwork and $\tau : n \to n$ is any map, then $N\tau$ denotes the n-dimensional Λ-hypernetwork with labellings defined by $N\tau(\bar{x}) = N(\tau(\bar{x}))$, for all $\bar{x} \in {}^{\leq n}n$.*

It is an easy exercise to check that $N\tau$ is indeed a hypernetwork. We can think of $N\tau$ as a hypernetwork that 'embeds' into N in the model-theoretic sense — the embedding is τ and it takes a node x of $N\tau$ to the node $\tau(x)$ of N. This embedding need not be one-one, of course, but it does preserve all labels.

Hyperbases are closed under substitutions (or non-injective embeddings):

Lemma 11 *Let $\tau : n \to n$ be a non-injective map. If $\mathcal{H}$ is any n-dimensional hyperbasis for $\mathcal{A}$, and $N \in \mathcal{H}$, then $N\tau \in \mathcal{H}$.*

PROOF:

Since τ is not injective, it is either a substitution $[i/j]$ or a product of substitutions (see, e.g., [24, corollary 1.2]). Hence it suffices to prove the lemma in the case $\tau = [i/j]$ for arbitrary $i, j < n$, which we may assume are distinct. This is straightforward and was proved in [8, lemma 38]. □

We can now prove:

LEMMA 12 *Let Λ be a non-empty set. If $\mathcal{H}$ is any n-dimensional Λ-hyperbasis over $\mathcal{A}$ then $\mathcal{H}\restriction_m$ is an m-dimensional Λ-hyperbasis over $\mathcal{A}$.*

PROOF:

We check the three conditions for hyperbases. If $a \in At(\mathcal{A})$ then there is $N \in \mathcal{H}$ with $N(0,1) = a$, so $N\restriction_m(0,1) = a$. For all $N\restriction_m$ with $N \in \mathcal{H}$, all $x, y, z < m$ with $z \neq x, y$, and all $a, b \in At(\mathcal{A})$ such that $N\restriction_m(x,y) \leq a\,;b$, we know $N(x,y) \leq a\,;b$, and since $\mathcal{H}$ is a hyperbasis, there is $N' \in \mathcal{H}$ with $N' \equiv_z N$, $N'(x,z) = a$, and $N'(z,y) = b$. The required m-dimensional hypernetwork is $N'\restriction_m$. Finally, if $N_1, N_2 \in \mathcal{H}$ and $x, y < m$ with $N_1\restriction_m \equiv_{xy} N_2\restriction_m$, then using $m \geq 3$, choose $z < m$ with $z \neq x, y$. By lemma 11, $N_1^* = N_1[m/z][m+1/z]\cdots[n-1/z]$ and $N_2^* = N_2[m/z]\cdots[n-1/z]$ belong to $\mathcal{H}$ and $N_1^* \equiv_{xy} N_2^*$. So there is $N_3 \in \mathcal{H}$ with $N_1^* \equiv_x N_3 \equiv_y N_2^*$. So $N_1\restriction_m \equiv_x N_3\restriction_m \equiv_y N_2\restriction_m$, as required. □

LEMMA 13 *Let $\mathcal{H}$ be any n-dimensional hyperbasis for $\mathcal{A}$. If $\bar{x} \in {}^{<\omega}n$ is any sequence such that $rg(\bar{x}) \neq n$, and $M, N \in \mathcal{H}$ satisfy $M \equiv_{\bar{x}} N$, then there exist hypernetworks $N_i \in \mathcal{H}$ for $i \leq |\bar{x}|$ such that $M = N_0$, $N = N_{|\bar{x}|}$ and $N_i \equiv_{x_i} N_{i+1}$ for $i < |\bar{x}|$.*

PROOF:

By induction on $|\bar{x}|$. If $|\bar{x}| \leq 2$ the result holds by definition of a hyperbasis. So let $\bar{x} \in {}^{<\omega}n$ be such that $rg(\bar{x}) \neq n$, $|\bar{x}| = l$ (say) > 2 and suppose the lemma holds for all shorter sequences. Pick $k \in n \setminus rg(\bar{x})$. Suppose $M, N \in \mathcal{H}$ and $M \equiv_{\bar{x}} N$. By lemma 11, $M[x_0/k], N[x_0/k] \in \mathcal{H}$, and $M[x_0/k] \equiv_{(x_1,x_2,\ldots,x_{l-1})} N[x_0/k]$. So inductively, there are hypernetworks $N_i \in \mathcal{H}$ for $1 \leq i \leq l$ such that

$$M \equiv_{x_0} M[x_0/k] = N_1 \equiv_{x_1} N_2 \;\cdots\; N_{l-1} \equiv_{x_{l-1}} N_l = N[x_0/k] \equiv_{x_0} N.$$

Finally we must reorder these equivalences. From the last two equivalences we see that $N_{l-1} \equiv_{x_0,x_{l-1}} N$. From the definition of a hyperbasis, there exists $N'_{l-1} \in \mathcal{H}$ with $N_{l-1} \equiv_{x_0} N'_{l-1} \equiv_{x_{l-1}} N$. Proceeding in this way, working down from $l-1$ to 1, we obtain hypernetworks $N'_i \in \mathcal{H}$ for $1 \leq i < l$ with

$$M \equiv_{x_0} N_1 \equiv_{x_0} N'_1 \equiv_{x_1} \cdots\; N'_{l-2} \equiv_{x_{l-2}} N'_{l-1} \equiv_{x_{l-1}} N.$$

Since the first two equivalences give $M \equiv_{x_0} N'_1$, we obtain our result. □

PROPOSITION 14 *Assume that Λ is a non-empty set and $\mathcal{G}, \mathcal{H}$ are m- and n-dimensional Λ-hyperbases over $\mathcal{A}$, respectively, such that $\mathcal{G} = \mathcal{H}\restriction_m$. Then $\mathfrak{Ca}(\mathcal{G})$ is isomorphic to a subalgebra of $\mathfrak{Nr}_m(\mathfrak{Ca}(\mathcal{H}))$.*

PROOF:

We define a map $h : \mathfrak{Ca}(\mathcal{G}) \to \mathfrak{Nr}_m(\mathfrak{Ca}(\mathcal{H}))$ by

$$h(S) = \{N \in \mathcal{H} : N\restriction_m \in S\}, \text{ for } S \subseteq \mathcal{G}.$$

It should be clear that $h(S)$ is indeed in the neat reduct: if $N \in h(S)$, $m \leq i < n$, and $N' \in \mathcal{H}$ with $N' \equiv_i N$, then $N'\restriction_m = N\restriction_m$ so $N' \in h(S)$. Hence, $\mathsf{c}_i(h(S)) = h(S)$, and $h(S) \in \mathfrak{Nr}_m(\mathfrak{Ca}(\mathcal{H}))$.

We show h is a cylindric algebra embedding. Let $S, S' \subseteq \mathcal{G}$. Clearly, $h(S \cup S') = h(S) \cup h(S')$. Because $\mathcal{H}\restriction_m = \mathcal{G}$, $h(\mathcal{G} \setminus S) = \mathcal{H} \setminus h(S)$. It follows that h preserves the boolean operations. Again by $\mathcal{H}\restriction_m = \mathcal{G}$, we see that if $h(S) = \emptyset$ then $S = \emptyset$, so that h is one-one.

For the diagonals, let $i, j < m$. Let $\mathsf{d}_{ij}^{\mathfrak{Ca}(\mathcal{G})}, \mathsf{d}_{ij}^{\mathfrak{Ca}(\mathcal{H})}$ denote the ijth diagonals of $\mathfrak{Ca}(\mathcal{G})$ and $\mathfrak{Ca}(\mathcal{H})$ respectively. Then $h(\mathsf{d}_{ij}^{\mathfrak{Ca}(\mathcal{G})}) = \{N \in \mathcal{H} : N\restriction_m(i,j) \leq 1'\} = \{N \in \mathcal{H} : N(i,j) \leq 1'\} = \mathsf{d}_{ij}^{\mathfrak{Ca}(\mathcal{H})}$.

Finally, for the cylindrifiers, let $S \subseteq \mathcal{G}$ and $i < m$; we show that $h(\mathsf{c}_i^{\mathfrak{Ca}(\mathcal{G})}(S)) = \mathsf{c}_i^{\mathfrak{Ca}(\mathcal{H})}(h(S))$. To do this, we require that for any $N \in \mathcal{H}$, $N\restriction_m \equiv_i P$ for some $P \in S$ if and only if $N \equiv_i Q$ for some $Q \in \mathcal{H}$ with $Q\restriction_m \in S$.

So let $N \in \mathcal{H}$. Right-to-left is straightforward: if $Q \in \mathcal{H}$, $N \equiv_i Q$, and $Q\restriction_m \in S$, then $N\restriction_m \equiv_i Q\restriction_m \in S$. For the converse, assume that $N\restriction_m \equiv_i P$ for some $P \in S$. As $\mathcal{G} = \mathcal{H}\restriction_m$, we may take $Q' \in \mathcal{H}$ satisfying $Q'\restriction_m = P$. Then, $N \equiv_{(i,m,m+1,\ldots,n-1)} Q'$. By lemma 13, there is $Q \in \mathcal{H}$ such that $N \equiv_i Q \equiv_{(m,m+1,\ldots,n-1)} Q'$. So $N \equiv_i Q$ and $Q\restriction_m = Q'\restriction_m = P \in S$, as required.

Thus, h is a cylindric algebra embedding. □

3 The relation algebras $\mathfrak{A}(n, r)$

Let $4 \leq n < \omega$ and $r < \omega$. $\mathfrak{A}(n, r)$ is a finite, symmetric algebra of the similarity type of relation algebras. Its atoms are: 1' (identity) and $a^k(i, j)$ for each $i < n-1$, $j < r$, and $k < \Omega$, where $n, r \ll \Omega < \omega$ — say, $\Omega = (nr)^{nr}$. All elements are self-converse. Next, we define composition by listing the *inconsistent triples* (a, b, c) of atoms of $\mathfrak{A}(n, r)$ — those such that $a \cdot (b\,;c) = 0$. This defines composition: for $x, y \in \mathfrak{A}(n, r)$ we have $x\,; y = \sum\{a \in At(\mathfrak{A}(n, r)) : \exists b, c \in At(\mathfrak{A}(n, r)),\ b \leq x,\ c \leq y,\ (a, b, c) \text{ is not inconsistent}\}$.

Any permutation of the triple $(1', s, t)$ will certainly be inconsistent unless $t = s$. Also, all permutations of the following triples are inconsistent:

$$(a^k(i, j), a^{k'}(i, j), a^{k''}(i, j')),$$

if $j \leq j' < r$, where $i < n-1$ and $k, k', k'' < \Omega$ are arbitrary. That is, a triple of a-atoms with all i-indices the same and two js the same must (for consistency) have the third j strictly less than the other two. All other triples of atoms are consistent. The superscripts k don't play much of a role here, but to prove that $\mathfrak{A}(n, r) \notin \mathsf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ (theorem 18) we will need 'many' atoms.

This completes the definition of $\mathfrak{A}(n, r)$. As abbreviations, we write

$$a(i, j) = \sum_{k<\Omega} a^k(i, j), \quad a(i) = \sum_{j<r} a(i, j), \quad a^k = \sum_{\substack{i<n-1\\ j<r}} a^k(i, j).$$

(All suprema exist because the algebra is finite.)

Note that as $\mathfrak{A}(n, r)$ is finite, it is isomorphic to the complex algebra over its atom structure, and also to its canonical extension $\mathfrak{A}(n, r)^+$. We'll see in corollary 16 that for $n \geq 4$, $\mathfrak{A}(n, r)$ is a relation algebra. Of course, $\mathfrak{A}(n, 0)$ has no atoms except 1' and so is the trivial representable relation algebra with domain $\{0, 1\}$, where $1' = 1$.

For $3 \leq m \leq n$, we also define an infinite algebra $\mathfrak{C}_r$ of the similarity type of CA_m by

$$\mathfrak{C}_r \stackrel{\text{def}}{=} \mathfrak{Ca}(H_m(\mathfrak{A}(n,r),\omega)).$$

(To avoid cluttering the notation, m and n here are determined by context.) We will see in corollary 17 that $\mathfrak{C}_r \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$.

3.1 $\mathfrak{A}(n,r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$

THEOREM 15 *Let $4 \leq n < \omega$, $r < \omega$, and let Λ be an arbitrary non-empty set. Then the set $H_n(\mathfrak{A}(n,r),\Lambda)$ (or $H_n(\mathfrak{A}(n,r))$ for short) of all n-dimensional Λ-hypernetworks over $\mathfrak{A}(n,r)$ is an n-dimensional Λ-hyperbasis for $\mathfrak{A}(n,r)$.*

PROOF:

We show that $H_n(\mathfrak{A}(n,r))$ is an n-dimensional Λ-hyperbasis. If $r=0$, all edge labels on any hypernetwork are 1'; under these circumstances it is easily seen that $H_n(\mathfrak{A}(n,r))$ is a hyperbasis. So we may suppose $r>0$.

First, let $a \in At\mathfrak{A}(n,r)$. Let N_a be the hypernetwork with edge labelling defined by: $N_a(x,1) = N_a(1,x) = a$ for $x < n$, $x \neq 1$, and all other edges are labelled by the identity 1'. Let the hyperedges of N_a be labelled by a constant $\lambda_0 \in \Lambda$, say. N_a is easily seen to be a hypernetwork, and so $N_a \in H_n(\mathfrak{A}(n,r))$. Thus, $H_n(\mathfrak{A}(n,r))$ satisfies the first requirement for hyperbases.

Second, let $N \in H_n(\mathfrak{A}(n,r))$, $x,y,z < n$, $z \neq x,y$, and $a,b \in At\mathfrak{A}(n,r)$, and suppose that $N(x,y) \leq a\,;b$. We seek a hypernetwork $M \equiv_z N$ with $M(x,z)=a$ and $M(z,y)=b$. If $a=1'$ then let $M=N[z/x] \in H_n(\mathfrak{A}(n,r))$ (see lemma 11). Evidently, $M \equiv_z N$, $M(x,z)=1'=a$, and $M(z,y)=N(x,y) \leq N(x,z)\,;N(z,y) = 1'\,;b = b$, so that $M(z,y)=b$ as $M(z,y)$ is an atom. Otherwise, if $b=1'$ then let $M=N[z/y]$; we see that $M \in H_n(\mathfrak{A}(n,r))$, $M(x,z)=a$, and $M(z,y)=b$ in a similar way.

So assume that $a,b \neq 1'$. We define M as follows. The condition $M \equiv_z N$ already defines all labels of edges and hyperedges of M not involving z; and we define $M(z,z)=1'$. We must define the labels on edges (w,z) in M, for $w<n$ with $w \neq z$ (hyperlabels will be dealt with later); because atoms of $\mathfrak{A}(n,r)$ are self-converse, by lemma 3 the converse edge (z,w) must have the same label as (w,z) in M, so we need only define one of them. These labels are defined one at a time, as follows. First we let $M(x,z)=a$ and $M(y,z)=b$; this is well-defined, as if $x=y$ then $N(x,y)=1' \leq a\,;b$ so that $a=b$. We continue through the remaining edges (w,z) in some arbitrary order, as follows. Let (w,z) be the next edge to label. If $N(w,v) \leq 1'$ for some $v<n$, $v \neq z$ such that (v,z) has already been labelled, then we have no choice but to let $N(w,z)=N(v,z)$. This is well-defined if there is more than one such v. If not, we let $M(w,z)=a^0(i,0)$, for some $i<n-1$ to be determined next. (We use $r>0$ here.) i is chosen as the least number such that there is no already-labelled edge (v_i,z) with $M(w,v_i), M(v_i,z) \leq a(i)$. Since there are only $n-2$ nodes v_i in the hypernetwork different from w and z, while there are $n-1$ possible values of i to choose from, it will always be possible to find such an i. It can be seen that for all $v,w \in n \setminus \{z\}$, the triple $(M(v,w), M(w,z), M(z,v))$ is consistent. Thus, we may define all the edge labels in M while avoiding any inconsistency. We still have to define the hyperlabels of hyperedges involving z in M. For this, observe that $M(w,z) \neq 1'$ for all $w < n$ with $w \neq z$. Thus, if $\bar{x}, \bar{y} \in {}^{\leq n}n$ and $z \in rg(\bar{x}) \setminus rg(\bar{y})$ then $\bar{x} \not\sim_M \bar{y}$. So we let $M(\bar{x}) = \lambda_0$ for all $\bar{x} \in {}^{\leq n}n$ with $z \in rg(\bar{x})$ and this must be consistent with the third part of the definition of a hypernetwork. Hence $M \in H_m(\mathfrak{A}(n,r))$.

Finally, for the last part of the definition of hyperbasis, let $M, N \in H_n(\mathfrak{A}(n,r))$ and $x, y < n$ be such that $M \equiv_{xy} N$. We seek a hypernetwork $L \in H_n(\mathfrak{A}(n,r))$ such that $M \equiv_x L \equiv_y N$. If $x = y$ then $L = M$ will do, so we may assume not. The requirement $M \equiv_x L \equiv_y N$ uniquely determines all the edge labels of L except the labels of (x,y) and (y,x). Even these are determined if $M(y,z) \leq 1'$ or $N(x,z) \leq 1'$ for some $z \in n \setminus \{x,y\}$, for then we must have $L = N[y/z]$ or $L = M[x/z]$, respectively; this can be checked to be well-defined if both occur, or if there are several such z. So we may assume not. The labels of (x,y), (y,x) are now defined to be $a^0(i,0)$ for some $i < n-1$ such that there is no node $v < n$, $v \neq x, y$, with $L(x,v), L(v,y) \leq a(i)$. Exactly as in the previous case, there will always be such a value i. As before, we let $L(\bar{x}) = \lambda_0$ for all $\bar{x} \in {}^{\leq n}n$ with $x, y \in rg(\bar{x})$. If $\bar{x}, \bar{y} \in {}^{\leq n}n$ and $\{x,y\} \subseteq rg(\bar{x})$ but $\{x,y\} \not\subseteq rg(\bar{y})$ then $\bar{x} \not\sim_L \bar{y}$, so the hyperlabelling is consistent and L is a hypernetwork.

Thus, $H_n(\mathfrak{A}(n,r))$ is an n-dimensional hyperbasis for $\mathfrak{A}(n,r)$, as claimed. $\square$

Corollary 16 *For $4 \leq n < \omega$, $r < \omega$, we have $\mathfrak{A}(n,r) \in \mathsf{RA}$ and $\mathfrak{A}(n,r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$.*

PROOF:

To show that $\mathfrak{A}(n,r)$ is a relation algebra, we refer to [15, theorem 2.2(5)]. Inspection of the definition of $\mathfrak{A}(n,r)$ and the proof of theorem 15 above shows that the hypotheses of this theorem are met; it follows that $\mathfrak{A}(n,r) \in \mathsf{RA}$. Now, by theorem 6, $\mathfrak{A}(n,r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. $\square$

Corollary 17 *Let $3 \leq m \leq n < \omega$, $n \geq 4$, and $r < \omega$. Then $\mathfrak{C}_r = \mathfrak{Ca}(H_m(\mathfrak{A}(n,r),\omega)) \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$.*

PROOF:

We saw that $H_n(\mathfrak{A}(n,r),\omega)$ was an n-dimensional ω-hyperbasis; so by proposition 8, $\mathfrak{Ca}(H_n(\mathfrak{A}(n,r),\omega)) \in \mathsf{CA}_n$. By lemma 12, $H_m(\mathfrak{A}(n,r),\omega)$ is an m-dimensional ω-hyperbasis, so $\mathfrak{Ca}(H_m(\mathfrak{A}(n,r),\omega)) \subseteq \mathfrak{Nr}_m\mathfrak{Ca}(H_n(\mathfrak{A}(n,r),\omega))$ by proposition 14. Thus, $\mathfrak{Ca}(H_m(\mathfrak{A}(n,r),\omega)) \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$. $\square$

3.2 $\mathfrak{A}(n,r) \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$

Theorem 18 *For $4 \leq n < \omega$ and $0 < r < \omega$, $\mathfrak{A}(n,r) \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$.*

PROOF:

By theorem 6 and the finiteness of $\mathfrak{A}(n,r)$, it is enough to establish that $\mathfrak{A}(n,r)$ ($\cong \mathfrak{A}(n,r)^+$) has no $(n+1)$-dimensional hyperbasis. So suppose for contradiction that $\mathcal{H}$ is such a hyperbasis. From this, we will show that for $t = 0, 1, \ldots, nr$ there is a 'large' set $S_t \subseteq \mathcal{H}$, where the hypernetworks in S_t are subject to certain constraints which depend on t and get tighter as t increases. This will lead to a contradiction when we take a sufficiently large value of t.

We begin by constructing S_0. Since $\mathcal{H}$ is a hyperbasis, there is a hypernetwork $N \in \mathcal{H}$ with $N(0,1) = a^0(0,0)$. Letting $\tau : (n+1) \to (n+1)$ be the map given by $\tau(1) = 1$, $\tau(i) = 0$ (all $i \leq n$ with $i \neq 1$), we see by lemma 11 that $N\tau \in \mathcal{H}$. For each $k < \Omega$ we have $a^k(0,0) ; a^0(1,0) \geq a^0(0,0)$ (check that this is not one of the inconsistent triples), so there is a hypernetwork $M_k \in \mathcal{H}$ with $N\tau \equiv_2 M_k$, $M_k(0,2) = a^k(0,0)$, and $M_k(2,1) = a^0(1,0)$. See figure 1.

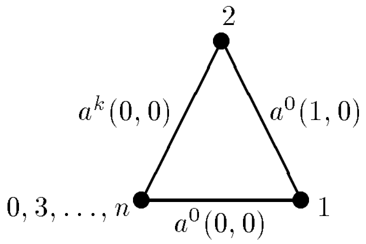

Figure 1: The hypernetwork M_k

Observe that $M_l \equiv_2 M_m$ for $l, m < \Omega$. Furthermore, if $M_l(0,2), M_m(0,2) \leq a^k$ (some $k < \Omega$) then as $M_l(0,2) \leq a^l$, $M_m(0,2) \leq a^m$, and $a^0, a^1, \ldots$ are pairwise disjoint in $\mathfrak{A}(n,r)$, we have $l = m$ $(= k)$. Let $S_0 = \{M_k : k < \Omega\}$.

Inductively, for $0 \leq t < nr$, suppose there is a set $S_t \subseteq \mathcal{H}$ of hypernetworks with

1. $|S_t| \geq (nr)^{nr-t}$,

an integer s_t with $1 \leq s_t \leq n$, and functions $I_t : (n+1) \setminus \{s_t\} \to (n-1)$, $J_t : (n+1) \setminus \{s_t\} \to r$, such that for any $L, M \in S_t$:

2. $L \equiv_{s_t} M$,
3. if $L(0, s_t), M(0, s_t) \leq a^k$ (any $k < \Omega$) then $L = M$,
4. $I_t(0) = J_t(0) = 0$,
5. $L(x, s_t) \leq a(I_t(x), J_t(x))$, for all $x \leq n$, $x \neq s_t$.

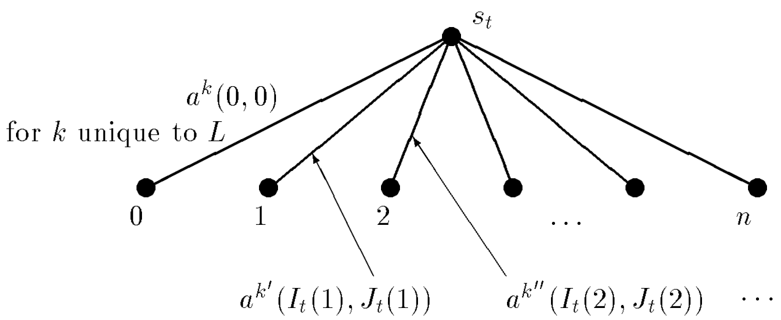

Figure 2: A hypernetwork $L \in S_t$

See figure 2. This is true for $t=0$ (with $s_0=2$) if we let $I_0(x)=0$ for $x\leq n, x\neq 1$, $I_0(1)=1$, and $J_0(x)=0$ for all $x\leq n$. See figure 1.

DEFINITION 19 Let $1\leq s\leq n$, $D=(n+1)\setminus\{s\}$, $I:D\to(n-1)$, and $J:D\to r$, as above. For $i<n-1$, define the *index of i with respect to I,J* by

$$Ind(i,I,J)=\min\{J(x):x\in D,\ I(x)=i\}-r<0$$

if the minimum is taken over a non-empty set, and let $Ind(i,I,J)=0$ otherwise.[2] Thus, $0\geq Ind(i,I,J)\geq -r$, for each $i<n-1$. Then define the *rank* of (I,J):

$$rk(I,J)=\sum_{i<n-1}Ind(i,I,J).$$

So $0\geq rk(I,J)\geq -(n-1)r$.

We have one more inductive assumption.

6. $rk(I_t,J_t)<-t$.

Observe that $Ind(0,I_0,J_0)=Ind(1,I_0,J_0)=-r$, and $Ind(i,I_0,J_0)=0$ for $i>1$. Hence, $rk(I_0,J_0)=-2r<0$. So all the hypotheses are true for $t=0$.

Assuming that S_t,s_t,I_t,J_t exist with properties 1 to 6, let us see if we can find $S_{t+1},s_{t+1},I_{t+1},J_{t+1}$ also with these properties.

Clearly, $I_t:(n+1)\setminus\{s_t\}\to(n-1)$ cannot be injective. Let $p,q\in(n+1)\setminus\{s_t\}$ be distinct such that $I_t(p)=I_t(q)$. We can suppose that $J_t(p)\geq J_t(q)$, and because $J_t(0)=0$, we can suppose further that $p>0$. *We will let $s_{t+1}=p$.* Observe that p does not contribute to the 'min' in the expression for $Ind(i,I_t,J_t)$ for $i=I_t(p)$, and certainly not for other i, so that for all $i<n-1$,

$$\begin{aligned} Ind(i,I_t,J_t) &= \begin{cases}\min(K_i)-r, & \text{if } K_i\neq\emptyset;\\ 0, & \text{otherwise,}\end{cases} && (4)\\ \text{where } K_i &= \{J_t(x):x\leq n, x\notin\{s_t,s_{t+1}\}, I_t(x)=i\}. && (5)\end{aligned}$$

Note that $1\leq s_{t+1}\leq n$ and $s_{t+1}\neq s_t$.

Fix $L_0\in S_t$. For each hypernetwork $M\in S_t\setminus\{L_0\}$, we have seen (lemma 11) that $M[s_{t+1}/s_t]\in\mathcal{H}$. Since $M\equiv_{s_t}L_0$, we have $M[s_{t+1}/s_t]\equiv_{s_t,s_{t+1}}L_0$. By the last property of hyperbases, we may choose $M'\in\mathcal{H}$ with

$$M[s_{t+1}/s_t]\ \equiv_{s_t}\ M'\ \equiv_{s_{t+1}}\ L_0. \qquad (6)$$

See figure 3. Let $S'_t=\{M':M\in S_t\setminus\{L_0\}\}\subseteq\mathcal{H}$. The required set S_{t+1} will be a subset of S'_t to be defined shortly. First, we show that properties 2 and 3 hold for S'_t and hence for any subset of it. Property 2 holds because $M'\equiv_{s_{t+1}}L_0$ for all $M'\in S'_t$. For property 3, observe that $M'(0,s_{t+1})=M[s_{t+1}/s_t](0,s_{t+1})=M(0,s_t)$ for any $M'\in S'_t$. If $M'_1(0,s_{t+1}),M'_2(0,s_{t+1})\leq a^k$ for some $M_1,M_2\in S_t\setminus\{L_0\}$, $k<\Omega$, then $M_1(0,s_t),M_2(0,s_t)\leq a^k$. Since $M_1,M_2\in S_t$, property 3 for S_t yields

[2] Or, as Roger Maddux puts it, "For an i-trip you pay your lowest ticket price; if you've not got a ticket you pay the maximum possible plus one".

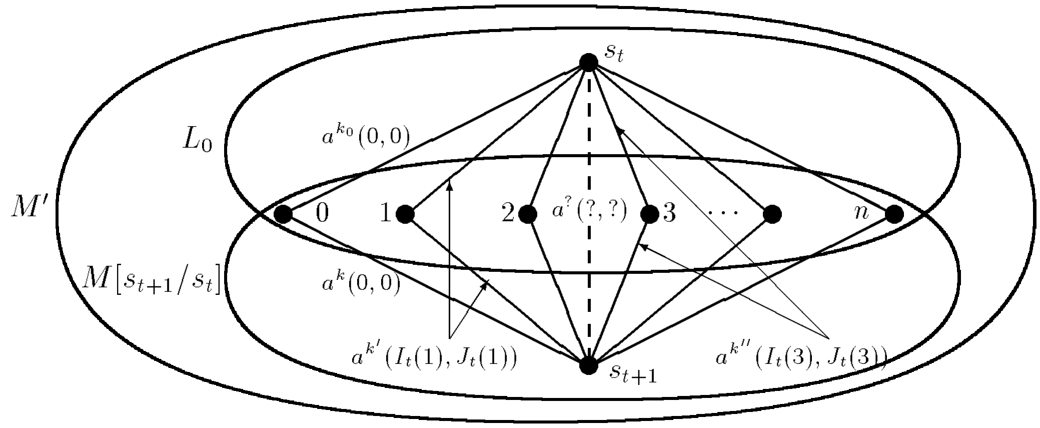

Figure 3: The hypernetwork M'

$M_1 = M_2$, so $M_1' = M_2'$. Note that by the same argument, $M_1' = M_2' \Rightarrow M_1 = M_2$, so by property 1 for S_t,

$$|S_t'| = |S_t| - 1. \tag{7}$$

We will find functions $I_{t+1} : (n+1) \setminus \{s_{t+1}\} \to (n-1)$, $J_{t+1} : (n+1) \setminus \{s_{t+1}\} \to r$ so that properties 1, 4, and 5 remain true for $S_{t+1}, s_{t+1}, I_{t+1}, J_{t+1}$, for some suitable $S_{t+1} \subseteq S_t'$. The domain of I_{t+1} is the same as that of I_t except that it includes s_t instead of s_{t+1}. We define I_{t+1} to agree with I_t on all points in both domains:

$$I_{t+1}(x) = I_t(x) \text{ for all } x \in dom(I_t) \cap dom(I_{t+1}) = (n+1) \setminus \{s_t, s_{t+1}\}, \tag{8}$$

and similarly for J_{t+1}. Now, only the values of $I_{t+1}(s_t)$, $J_{t+1}(s_t)$ remain undecided: they will be settled below.

Property 4 for I_{t+1}, J_{t+1} is obvious, given that $0 \in dom(I_t) \cap dom(I_{t+1})$. The case where $x \neq s_t$ in property 5 will automatically hold, because for any $M \in S_t \setminus \{L_0\}$, $M'(x, s_{t+1}) = M[s_{t+1}/s_t](x, s_{t+1}) = M(x, s_t) \leq a(I_t(x), J_t(x)) = a(I_{t+1}(x), J_{t+1}(x))$ by (6), (8), and property 5 for S_t.

But the case $x = s_t$ in property 5 is not covered, as we do not know much about $M'(s_t, s_{t+1})$ for $M \in S_t \setminus \{L_0\}$. At least, we can show that it is disjoint from 1'. For this, note that $M'(0, s_t) = L_0(0, s_t)$, and $M'(0, s_{t+1}) = M[s_{t+1}/s_t](0, s_{t+1}) = M(0, s_t)$. Since $L_0 \neq M$, property 3 implies that $L_0(0, s_t) \neq M(0, s_t)$. Hence, $M'(0, s_t) \neq M'(0, s_{t+1})$, so because M' is a hypernetwork, $M'(s_t, s_{t+1}) \neq 1'$.

Thus, for each $M' \in S_t'$ we have $M'(s_t, s_{t+1}) \leq a(i, j)$ for some i and j which depend on M'. The number of possible values of (i, j) is at most $(n-1)r$. Using (7), property 1 for S_t, and the pigeon-hole principle, we can find a subset S_{t+1} of S_t' with

$$|S_{t+1}| \geq \frac{|S_t'|}{(n-1)r} \geq \frac{(nr)^{nr-t} - 1}{(n-1)r} \geq (nr)^{nr-(t+1)},$$

such that there are fixed $i_0 < n-1$ and $j_0 < r$ with $M'(s_t, s_{t+1}) \leq a(i_0, j_0)$ for all $M' \in S_{t+1}$. We now complete the definition of the functions I_{t+1} and J_{t+1} by: $I_{t+1} : s_t \mapsto i_0$ and $J_{t+1} : s_t \mapsto j_0$.

We now have $S_{t+1}, s_{t+1}, I_{t+1}, J_{t+1}$. By construction, property 5 holds even for the case $x = s_t$. Since we have chosen S_{t+1} to be 'large', property 1 holds too.

Finally, we check property 6. As this property holds for t, it is enough to prove that $rk(I_{t+1}, J_{t+1}) < rk(I_t, J_t)$. We do this by showing that $Ind(i, I_{t+1}, J_{t+1}) \leq Ind(i, I_t, J_t)$ for all $i < n-1$, and that $Ind(i_0, I_{t+1}, J_{t+1}) < Ind(i_0, I_t, J_t)$. (This can also be seen informally by inspecting figure 3.)

Recall the definition of Ind:

$$Ind(i, I_{t+1}, J_{t+1}) \quad = \quad \min\{J_{t+1}(x) : x \leq n, x \neq s_{t+1}, I_{t+1}(x) = i\} - r \tag{9}$$

if this set is non-empty, and 0 otherwise. By (8), we can replace I_t, J_t by I_{t+1}, J_{t+1} in (5), without changing K_i. So $K_i = \{J_{t+1}(x) : x \leq n, x \notin \{s_t, s_{t+1}\}, I_{t+1}(x) = i\}$, and by (9),

$$Ind(i, I_{t+1}, J_{t+1}) \quad = \quad \min(K_i \cup \{J_{t+1}(s_t) : I_{t+1}(s_t) = i\}) - r \tag{10}$$

if the set is non-empty, and 0, otherwise. Comparing (4) and (10), it is plain that $Ind(i, I_{t+1}, J_{t+1}) \leq Ind(i, I_t, J_t)$, since the minimum in (10) is taken over a larger set.

We now prove that $Ind(i_0, I_{t+1}, J_{t+1}) < Ind(i_0, I_t, J_t)$. We defined $I_{t+1}(s_t) = i_0$ and $J_{t+1}(s_t) = j_0$, so (10) reduces to:

$$Ind(i_0, I_{t+1}, J_{t+1}) = \min(K_{i_0} \cup \{j_0\}) - r. \tag{11}$$

Comparing (11, 4), it is clear that $Ind(i_0, I_{t+1}, J_{t+1}) < Ind(i_0, I_t, J_t)$ if $K_{i_0} = \emptyset$. If $K_{i_0} \neq \emptyset$, we require $j_0 < \min(K_{i_0})$. So take an arbitrary element of K_{i_0}: it has the form $J_t(x)$, where $x \leq n$, $x \neq s_t, s_{t+1}$, and $I_t(x) = i_0$. We show $j_0 < J_t(x)$. Now by property 1, $S_{t+1} \neq \emptyset$. Let $M' \in S_{t+1}$, where $M \in S_t \setminus \{L_0\}$. Then

$$\begin{aligned}
&M'(x, s_t) = L_0(x, s_t) \leq a(I_t(x), J_t(x)) = a(i_0, J_t(x)),\\
&M'(x, s_{t+1}) = M[s_{t+1}/s_t](x, s_{t+1}) = M(x, s_t) \leq a(I_t(x), J_t(x)) = a(i_0, J_t(x)),\\
&M'(s_t, s_{t+1}) \leq a(i_0, j_0).
\end{aligned}$$

As M' is a hypernetwork, this implies that $[a(i_0, J_t(x))\,; a(i_0, J_t(x))] \cdot a(i_0, j_0) \neq 0$. From the definition of composition in $\mathfrak{A}(n, r)$, this gives $j_0 < J_t(x)$, as required.

Thus, for $t < nr$, S_t, s, I_t, J_t exist with the listed properties. Now, taking $t = (n-1)r$ would give us a set $S_{(n-1)r}$ of size at least $(nr)^{nr-(n-1)r}$ — certainly non-empty — where the rank of all the hypernetworks would be constrained by $rk(I_t, J_t) < -(n-1)r$. But by definition 19, the minimum possible rank is just $-(n-1)r$. From this contradiction we deduce that the hyperbasis $\mathcal{H}$ does not exist. Hence, $\mathfrak{A}(n, r) \notin \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. $\square$

Recall that $\mathfrak{C}_r = \mathfrak{Ca}(H_m(\mathfrak{A}(n, r), \omega))$ for $r < \omega$, where $H_m(\mathfrak{A}(n, r), \omega)$ is the set of all m-dimensional ω-hypernetworks over $\mathfrak{A}(n, r)$. We have seen (corollary 17) that $\mathfrak{C}_r \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$.

COROLLARY 20 *If $4 \leq n < \omega$, $3 \leq m \leq n$, and $0 < r < \omega$, then $\mathfrak{C}_r \notin \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$.*

PROOF:

Suppose, for contradiction, that $\mathfrak{C}_r \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$. Then we have $\mathfrak{Ra}(\mathfrak{C}_r) \in \mathfrak{Ra}\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1} \subseteq \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. But it is easy to check that $\mathfrak{A}(n, r) \subseteq \mathfrak{Ra}(\mathfrak{C}_r)$ via the embedding $a \mapsto \{M \in H_m(\mathfrak{A}(n, r), \omega) : M(0, 1) \leq a\}$. So $\mathfrak{A}(n, r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$, contrary to theorem 18. $\square$

4 Games

Here, we introduce the games mentioned in the introduction. Our main result is theorem 24 below.

Definition 21 Let $3 \leq m \leq n < \omega$, $r \leq \omega$, let $\mathcal{A}$ be an atomic relation algebra, and let Λ be any non-empty set. We define a game $G_r^{m,n}(\mathcal{A}, \Lambda)$ (or we may just write $G_r^{m,n}(\mathcal{A})$ if Λ is given in the context), played by two players, $\forall$ (male) and $\exists$ (female), on n-dimensional Λ-hypernetworks. This game is designed so that for countable $\mathcal{A}$, a winning strategy for $\exists$ ensures that $\mathcal{A}$ has an n-dimensional Λ-hyperbasis and so $\mathcal{A} \in \mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$. But the game is designed to be stronger than that: a winning strategy for $\exists$ over the countable algebra $\mathcal{A}$, with a countable set Λ of labels, actually yields an n-dimensional Λ-hyperbasis such that any m-dimensional Λ-hypernetwork can be extended to an n-dimensional Λ-hypernetwork in the hyperbasis. This additional strength is required for us to prove the non-finite axiomatisability of $\mathsf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ over $\mathsf{S}\mathfrak{Nr}_m\mathsf{CA}_n$, for $3 \leq m < n$.

The rounds are numbered $0, 1, \ldots, t, \ldots$ for $t < r$, and in each round, $\exists$ produces an n-dimensional hypernetwork N_t, say, in response to some action by $\forall$. In round t:

1. $\forall$ may pick any m-dimensional hypernetwork M: i.e., he can freely choose all edge labels and hyperlabels of M subject to the result being a hypernetwork. We call such a move an *m-dimensional move* and denote it by (M). $\exists$ must respond with an n-dimensional hypernetwork N_t such that $N_t\restriction_m = M$.

2. $\forall$ may pick a previously played hypernetwork N_u (some $u < t$), nodes $x, y, z < n$ with $z \neq x, y$, and atoms $a, b \in At(\mathcal{A})$ such that $N_u(x, y) \leq a\,;b$. We call such a move by $\forall$ a *triangle move* and denote it by (N_u, x, y, z, a, b). $\exists$ must respond with a hypernetwork N_t such that $N_t \equiv_z N_u$ and $N_t(x, z) = a$ and $N_t(z, y) = b$.

3. Or he may choose (M, N, x, y), where $M, N \in \{N_u : u < t\}$ and $x, y < n$ are distinct, such that $M \equiv_{xy} N$. Such a move by $\forall$ is called an *amalgamation* move, denoted (M, N, x, y). $\exists$ must then respond with a hypernetwork N_t satisfying $N \equiv_x N_t \equiv_y M$. See figure 4; here we write $N - x$ for the 'hypernetwork' resulting from N by deleting node x, and define $M - y$ similarly.

$\exists$ wins the play if she never gets stuck in any round. As a convention, we say that $\exists$ always wins the game $G_0^{m,n}(\mathcal{A}, \Lambda)$.

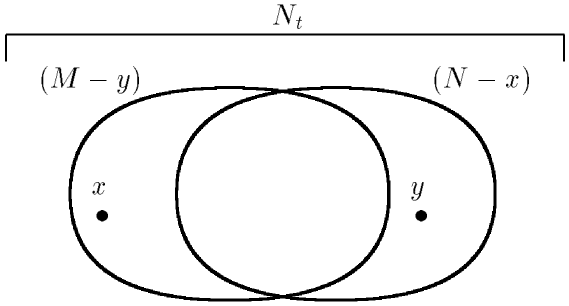

Figure 4: An amalgamation move

A *strategy* for $\exists$ in $G^{m,n}_r(\mathcal{A},\Lambda)$ is a set of rules telling her how to respond to any move that $\forall$ may make in any round. The advice given by the strategy in some round will tell her which hypernetwork to play; it may offer her more than one choice, but it should always suggest at least one. The strategy is said to be *winning* if it always suggests legal responses and $\exists$ wins any play of the game in which she uses it throughout.

PROPOSITION 22 *Let $3 \leq m \leq n < \omega$, let $\mathcal{A}$ be any atomic relation algebra, and let Λ be a non-empty set.*

1. *If $\mathcal{A}$ has an n-dimensional Λ-hyperbasis $\mathcal{H}$ such that $\mathcal{H}{\restriction}_m = H_m(\mathcal{A},\Lambda)$, then $\exists$ has a winning strategy in $G^{m,n}_\omega(\mathcal{A},\Lambda)$.*

2. *If $\mathcal{A}$ has countably many atoms, Λ is countable, and $\exists$ has a winning strategy in the game $G^{m,n}_\omega(\mathcal{A},\Lambda)$ then $\mathcal{A}$ has an n-dimensional Λ-hyperbasis $\mathcal{H}$ with $\mathcal{H}{\restriction}_m = H_m(\mathcal{A},\Lambda)$.*

PROOF:

If $\mathcal{A}$ has a hyperbasis $\mathcal{H}$ with extensions of arbitrary m-dimensional hypernetworks then $\exists$'s strategy is to always respond with a suitable hypernetwork from $\mathcal{H}$.

For the second part, assume that $\exists$ has a winning strategy, Λ is countable, and $\mathcal{A}$ has countably many atoms. Suppose that during a play of $G^{m,n}_\omega(\mathcal{A},\Lambda)$, $\forall$ plays (i) the m-dimensional move (M), for each m-dimensional Λ-hypernetwork M, (ii) the triangle move (N,x,y,z,a,b) for all hypernetworks N that occur in this play of the game and all legitimate x,y,z,a,b, and (iii) the amalgamation move (M,N,x,y) for all hypernetworks M,N that are played and all distinct $x,y < n$ such that $M \equiv_{xy} N$. Since there are only countably many atoms in $\mathcal{A}$ and countably many labels in Λ, it is possible to schedule all these moves in a play of the game. Let $\mathcal{H} = \{N : N \text{ occurs in the play}\}$. Clearly, $\mathcal{H}$ is an n-dimensional hyperbasis for $\mathcal{A}$ with an extension of any m-dimensional Λ-hypernetwork. $\Box$

REMARK 23 If we change the game $G^{m,n}_\omega(\mathcal{A},\Lambda)$ to another game, $R^{m,n}_\omega(\mathcal{A},\Lambda)$, in which $\forall$ is only allowed to make m-dimensional moves and triangle moves (no amalgamation moves), we obtain the following results: for any atomic relation algebra $\mathcal{A}$ with countably many atoms, if $\exists$ has a winning strategy in $R^{m,n}_\omega(\mathcal{A},\Lambda)$ for some Λ then she has a winning strategy in the game $R^{m,n}_\omega(\mathcal{A},\Lambda')$ for all non-empty sets Λ'; and if she has such a winning strategy then $\mathcal{A}$ has an n-dimensional relational basis (see [16, 10]).

4.1 $\exists$ has a winning strategy in $G^{m,n+1}_r(\mathfrak{A}(n,r))$

THEOREM 24 *Let $\mathfrak{A}(n,r)$ be a relation algebra as defined in section 3. Let $3 \leq m < n < \omega$ and $r < \omega$, and let Λ be an infinite set. Then $\exists$ has a winning strategy in $G^{m,n+1}_r(\mathfrak{A}(n,r),\Lambda)$.*

PROOF:

Certainly, $\exists$ has a winning strategy in $G^{m,n+1}_0(\mathfrak{A}(n,0))$, as no rounds are played. So we suppose $r \geq 1$. All hypernetworks mentioned will be Λ-hypernetworks. Recall that given an $(n+1)$-dimensional hypernetwork N with nodes $\{0,\ldots,n\}$, $\sim_N$ is the equivalence relation on ${}^{\leq n+1}(n+1)$ defined by $\bar{x} \sim_N \bar{y}$ if and only if $|\bar{x}| = |\bar{y}| = l$ (say) and $N(x_i, y_i) \leq 1'$ for all $i < l$. Requirement 3 in the definition of hypernetworks says that $\bar{x} \sim_N \bar{y} \Rightarrow N(\bar{x}) = N(\bar{y})$. Observe that the relation $\sim_N$ is determined by the edge labelling part of N alone.

We define a strategy for $\exists$ in $G_r^{m,n+1}(\mathfrak{A}(n,r), \Lambda)$. Let the play so far be $N_0, N_1, \ldots, N_{t-1}$. In round t, whatever type of move $\forall$ makes, $\exists$ has to choose a suitable hypernetwork N_t. We will specify below how $\exists$ labels the edges of N_t. This determines the relation $\sim_{N_t}$. But having done this, there is an easy way to specify the labels on hyperedges of N_t, which $\exists$ uses on many occasions. We will call this method the *default labelling.* It is as follows.

Some of the labels may be determined by $\forall$'s move. For example, if he plays an amalgamation move (L, M, x, y) in round t then $\exists$ must reply with a hypernetwork N_t such that $L \equiv_x N_t \equiv_y M$. So if $\bar{x} \in {}^{\leq n+1}((n+1) \setminus \{x\})$ then $N_t(\bar{x}) = L(\bar{x})$, etc. The label on such a hyperedge, and any $\sim_{N_t}$-equivalent hyperedge, is determined directly by $\forall$'s move. $\exists$ labels all the other hyperedges one at a time, as follows. If $\bar{x}$ is the next hyperedge to be labelled, then:

1. If $\bar{x} \sim_{N_t} \bar{y}$ for some hyperedge $\bar{y}$ that is already labelled in N_t, she lets $N_t(\bar{x}) = N_t(\bar{y})$.
2. Otherwise, she lets $N_t(\bar{x})$ be some new label that is not the label of any hyperedge that is already labelled in N_t nor is it the label of any hyperedge in any hypernetwork N_u for $u < t$. We do need Λ to be either infinite or at any rate fairly large for this to be possible. *In this case we say that $\exists$ chose the label $N_t(\bar{x})$ in round t.*

Thus, if $\exists$ chose the label $N_t(\bar{x})$ in round t then for all $s \leq t$ and for all $\bar{y} \in {}^{\leq n+1}(n+1)$ we have $N_t(\bar{x}) = N_s(\bar{y}) \Leftrightarrow (s = t) \wedge (\bar{x} \sim_{N_t} \bar{y})$.

$\exists$'s strategy in round t. If $\forall$ makes an m-dimensional move (type 1) in round t, and chooses an m-dimensional hypernetwork M over $\mathfrak{A}(n,r)$, $\exists$ lets N_t be any $(n+1)$-dimensional hypernetwork such that $N_t\restriction_m = M$ and $N_t(j, 0) = 1'$ for $m \leq j < n+1$. As $(n+1)$-tuples are not labelled in M, N_t is not unique, but it is easy to see that some such N_t exists, for example by using the default labelling. Observe that because $m < n$,

$$\text{there do not exist } i_0, \ldots, i_{n-1} < n+1 \text{ such that the atoms } N_t(i_{n-1}, i_j) \text{ (for } j = 0, \ldots, n-2) \text{ are pairwise distinct and not equal to } 1'. \tag{12}$$

This is the only kind of move that $\forall$ can make in $G_1^{m,n+1}(\mathfrak{A}(n,1))$, so the proof is complete for that case. From now on, we suppose $r \geq 2$.

Suppose that $\forall$ makes a triangle move of type 2: say (N_u, x, y, z, a, b) for $u < t$, $x, y, z \leq n$, $z \notin \{x, y\}$, and $a, b \in At(\mathfrak{A}(n,r))$ with $N_u(x,y) \leq a\,;b$. $\exists$ must find $N_t \equiv_z N_u$ with $N_t(x,z) = a$ and $N_t(z,y) = b$. If $a = 1'$ then she must let $N_t = N_u[z/x]$, and if $b = 1'$ she must let $N_t = N_u[z/y]$. This is well-defined if $a = b = 1'$. Assume now that $a, b \neq 1'$. $\exists$ first labels the edges of N_t.

Her technique is similar to that of theorem 15, but as the dimension is now $n+1$, there is an extra node to worry about. Since she must arrange that $N_t \equiv_z N_u$, the only edges she has to label are those involving z. She enumerates the n edges

(w, z) $(w \leq n,\ w \neq z)$ in some fashion, with $(x, z), (y, z)$ first, and labels them one by one. (The converse edges are given the same labels, as usual.) She lets $N_t(x, z) = a$ and $N_t(y, z) = b$; this is well-defined if $x = y$, since then, $a = b$.

She continues as follows. Let (w, z) be the next edge to label. If $N_u(w, v) = 1'$ for some edge (v, z) that has already been labelled, she must define $N_t(w, z) = N_t(v, z)$; this is well-defined. Otherwise, if (w, z) is not the final (nth) edge in the enumeration, she lets $N_t(w, z) = a^0(i, r-1)$ for some $i < n-1$ such that every previously-labelled edge (v, z), including (x, z) and (y, z), satisfies $N_t(v, z) \not\leq a(i)$. At this stage, she has labelled at most $n-2$ previous edges, and there are $n-1$ possible values of i to choose from, so this can be done. Finally, if (w, z) is the last, nth edge to label, she lets $N_t(w, z) = a^0(i, j)$ for some $i < n-1$ such that no already-labelled edge (v, z) *except perhaps the* $(n-1)$*th edge,* (w', z)*, say,* has label beneath $a(i)$ (again there are enough values of i to find a solution), and $j = r-1$ *unless* $N_u(w, w'), N_t(w', z) \leq a(i, r-1)$ *in which case* $j = r-2 \geq 0$. By choice of i, the triangle (w, v, z) of N_t is consistent, for $v \neq w'$. Consider the triangle (w, w', z).

1. If $N_u(w, w'), N_t(w', z) \leq a(i, r-1)$, then $\exists$ chose $N_t(w, z) = a^0(i, r-2)$, so (w, w', z) is certainly consistent.
2. Otherwise, if $N_u(w, w'), N_t(w', z) \leq a(i)$, then as $n \geq 4$ we have $w' \neq x, y$, so that $\exists$ chose $N_t(w', z) = a^0(i, r-1)$ a moment ago. So as case 1 fails, we must have $N_u(w, w') \leq a(i, l)$ for some $l < r-1$. But $\exists$ chose $N_t(w, z) = a^0(i, r-1)$, so again, (w, w', z) is consistent.
3. Otherwise, $N_u(w, w') \not\leq a(i)$ or $N_t(w', z) \not\leq a(i)$, so consistency is clear.

$\exists$ then sets $N_t(z, z) = 1'$, and uses the default labelling for hyperedges. Labels of hyperedges not involving z are determined directly by N_u, and other hyperedges are labelled with a unique new label as described above. It is plain that the resulting N_t is a hypernetwork.

Remark 25 The proof so far covers m-dimensional moves and triangle moves by $\forall$ and does not depend on t or Λ. Effectively, we have proved that $\exists$ has a winning strategy in the game $R^{m,n+1}_{\omega}(\mathfrak{A}(n, r), \Lambda)$ mentioned in remark 23. Consequently, for $r \geq 1$, $\mathfrak{A}(n, r)$ has an $(n+1)$-dimensional relational basis and so is in RA_{n+1}, but is not in $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. So $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n \subset \mathsf{RA}_n$ for all finite $n \geq 5$.

Problem 1 *For which $m > n \geq 4$ does $\mathfrak{A}(n, r)$ have an m-dimensional relational basis?* ($\mathsf{RA}_m \setminus \mathsf{S}\mathfrak{Ra}\mathsf{CA}_{n+1} \neq \emptyset$ *for any such* m.)

Suppose finally that $\forall$ plays an amalgamation move, say (M, N, x, y), in round t, for $M, N \in \{N_u : u < t\}$ with $M \equiv_{xy} N$ for distinct $x, y < n+1$. See figure 5. $\exists$'s strategy should select a hypernetwork N_t such that $M \equiv_x N_t \equiv_y N$.

$\exists$'s strategy (for rounds when $\forall$ makes an amalgamation move) is as follows. There are two cases, according to whether $M - x$ and $N - y$ embed in the sense of definition 10 into a hypernetwork N_u played earlier, the two embeddings agreeing up to $\sim_{N_u}$-equivalence on the 'common part' of M, N.

1. If there are $u < t$ and maps $\xi_1, \xi_2 : (n+1) \to (n+1)$ such that $M \equiv_x N_u\xi_1$, $N \equiv_y N_u\xi_2$ (see definition 10), and $N_u(\xi_1(z), \xi_2(z)) = 1'$ for all $z \in (n+1) \setminus$

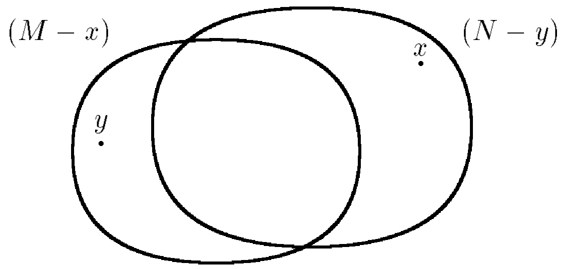

Figure 5: $\forall$'s move in round t

$\{x, y\}$, then she chooses any such ξ_1, ξ_2, u and uses them to define N_t. More precisely, let $\xi : n+1 \to n+1$ be defined by $\xi(x) = \xi_2(x)$, and $\xi(z) = \xi_1(z)$ if $z \neq x$. Then N_t is defined to be $N_u\xi$. It is easy to check that $M \equiv_x N_t \equiv_y N$.

2. Otherwise, she selects an atom $a^0(i, j)$ to label (x, y) such that:

   (a) Every triangle of N_t of the form (x, y, z) is consistent — i.e., there is no $z \leq n$, $z \neq x, y$, such that the triple $(a^0(i, j), M(y, z), N(z, x))$ violates the rules defining $\mathfrak{A}(n, r)$ given at the start of section 3.

   (b) $r - 1 - t \leq j < r$.

   She then chooses hyperlabels using the default labelling in the way described before. Notice that if there is $z \leq n$ with $z \neq x, y$ such that $N(x, z) = 1$', then case 1 can be used, by taking $u < t$ with $N_u = M$ and letting $\xi_1 = Id_{(n+1)}$ and $\xi_2 = [x/z]$. The situation if $M(y, z) = 1$' for some z is similar. So neither is the case. Hence, the default labelling will choose new labels for all hyperedges involving both x and y.

This completes the definition of $\exists$'s strategy for amalgamation moves. We check that it can always be implemented. We assume inductively that she has successfully implemented it, and the other strategies for m-dimensional and triangle moves given above, in all rounds $u < t$. Let $\forall$ make the amalgamation move (M, N, x, y) as above in round t. First off, note that $t > 0$, else $\forall$ could not find M, N. If $t = 1$, then $M = N = N_0$, so $\exists$ can respond using case 1 of her strategy (with $u = 0$, $\xi_1 = \xi_2 = Id_{n+1}$). So we may assume that $t \geq 2$. As already mentioned, if there is $z \in (n+1) \setminus \{x, y\}$ with $N(x, z) \leq 1$' or $M(y, z) \leq 1$', then case 1 applies; so we may assume not.

Suppose that case 2 of $\exists$'s strategy cannot be used, because no suitable atom $a^0(i, j)$ for any $i < n - 1$ with $j \geq r - 1 - t$ can be found to label $N_t(x, y)$. We will show that case 1 of the strategy can be used in this situation.

By inapplicability of case 2, for each $i < n-1$ there is $z_i \in (n+1) \setminus \{x, y\}$ such that the triple $(a^0(i, r-1-t), M(y, z_i), N(z_i, x))$ is inconsistent. Recall that the only inconsistent triples are those consisting of 1' and two distinct atoms, or triples of three $a^k(i, j)$s with all three is the same and two js the same and no bigger than ($\leq$) the third j. We assumed above that $N(x, z_i), M(z_i, y) \not\leq 1$'. So inconsistency of $(a^0(i, r-1-t), M(y, z_i), N(z_i, x))$ implies that $M(y, z_i), N(z_i, x) \leq a(i)$. Since the $a(i')$ $(i' < n - 1)$ are pairwise disjoint in $\mathfrak{A}(n, r)$, the $z_{i'}$ must be pairwise distinct. So *for any* $j \geq r - 1 - t$, the triple $(a^0(i, j), M(y, z_{i'}), N(z_{i'}, x))$ is certainly consistent if $i' \neq i$. But by cardinalities, $n+1 \setminus \{x,y\} = \{z_i : i < n-1\}$, so $(a^0(i,j), M(y,z), N(z,x))$ is consistent for all $z \in (n+1) \setminus \{x,y,z_i\}$. We conclude that for every $j \geq r-1-t$, the triple $(a^0(i,j), M(y,z_i), N(z_i,x))$ is inconsistent. Now if $N(x,z_i) \leq a(i,j)$ and $M(z_i,y) \leq a(i,j')$ for distinct $j,j' < r$, then $(a^0(i,r-1), M(y,z_i), N(z_i,x))$ is consistent. So we must have $M(y,z_i), N(z_i,x) \leq a(i,j_i)$ for some single $j_i < r$; and $j_i \leq r-1-t$, else $(a^0(i,r-1-t), M(y,z_i), N(z_i,x))$ is consistent.

So we have arrived at the following relatively simple situation: for each $i < n-1$, there are $z_i \in (n+1) \setminus \{x,y\}$ and $j_i \leq r-1-t$ such that $M(y,z_i), N(z_i,x) \leq a(i,j_i)$. The z_i are pairwise distinct, and $\{x,y,z_0,\ldots,z_{n-2}\} = n+1$. See figure 6.

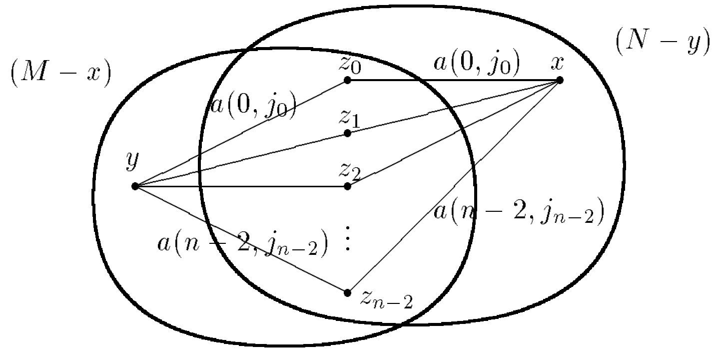

Figure 6: $\forall$'s move in round t in more detail

Choose least possible $u_1 < t$ such that there is a map $\xi_1 : (n+1) \to (n+1)$ with $M \equiv_x N_{u_1}\xi_1$ (informally, $M-x$ embeds into N_{u_1}). Since $M \in \{N_u : u < t\}$, such a u_1 exists. Similarly, choose $\xi_2 : (n+1) \to (n+1)$ such that $N \equiv_y N_{u_2}\xi_2$ for least possible $u_2 < t$. Define $\bar{z} = (z_0,\ldots,z_{n-2}) \in {}^{\leq n+1}(n+1)$. We will show that $u_1 = u_2$ and $\xi_1(\bar{z}) \sim_{N_{u_1}} \xi_2(\bar{z})$, so that case 1 of $\exists$'s strategy applies.

Consider u_1. We aim now to show that $\exists$ *chose the label on the hyperedge* $\xi_1(\bar{z})$ *of* N_{u_1} *in round* u_1 *of the game, using the default labelling.* To do this, we deal in turn with each kind of move that $\forall$ may have made in that round.

m-dimensional moves First, there are at least $n-1$ distinctly labelled edges in M incident with a common node: to wit, $(y,z_0),\ldots,(y,z_{n-2})$, and none of these edges is labelled by the identity (see figure 6). As x is not involved in these edges and $M \equiv_x N_{u_1}\xi_1$, the same holds for $N_{u_1}\xi_1$ and hence for N_{u_1}. By $\exists$'s earlier strategy (see (12) above) and since $m \leq n-1$, this cannot be so if she was responding to an m-dimensional move by $\forall$ in round u_1. So he did not make such a move. *The assumption $m < n$ is vital here, but only here.*

Triangle moves Assume that $\forall$ made the triangle move (N_v, x', y', z', a, b) in round u_1. So $v < u_1$. It follows by minimality of u_1 that $z' \in rg(\xi_1 \restriction_{(n+1)\setminus\{x\}})$. (For if not, then since $N_v \equiv_{z'} N_{u_1}$, we have $N_v\xi_1 \equiv_x N_{u_1}\xi_1 \equiv_x M$ and we could have taken $u_1 = v$.) We claim that $z' \neq \xi_1(y)$. For suppose otherwise. Recall that $t \geq 2$. We know (by $M \equiv_x N_{u_1}\xi_1$ and figure 6) that for each

$i < n-1$,

$$N_{u_1}(z', \xi_1(z_i)) = N_{u_1}\xi_1(y, z_i) = M(y, z_i) \leq a(i, j_i), \\ \text{where } \ j_i \leq r-1-t \leq r-3.$$

The $a(i, j_i)$ for $i < n-1$ are pairwise disjoint, and $n \geq 4$, so there are at least three different values of $N_{u_1}(z', \xi_1(z_i))$ as i varies. Now by minimality of u_1, $N_{u_1} \neq N_v[z'/x'], N_v[z'/y']$ (else $M \equiv_x N_v[z'/x']\xi_1$, etc). By examining $\exists$'s strategy in triangle moves, we see that in round u_1 $\exists$ must have chosen all but at most two of the edge labels (z', w) for $w \neq z'$. So she chose at least one of the edge labels $N_{u_1}(z', \xi_1(z_i))$. But her strategy only chooses atoms $a^0(i, j)$ for $j = r-1$ or $j = r-2$. As $j_i \leq r-3$, this is a contradiction, proving the claim.

Hence, $z' = \xi_1(z_i)$ for some $i < n-1$. Looking again at the default labelling method now shows that $\exists$ chose the label $N_{u_1}(\xi_1(\bar{z}))$, as the 'new node' z' of N_{u_1} is one of the nodes in this hyperedge.

Amalgamation moves Suppose now that $\forall$ made an amalgamation move in round u_1 — say, (M', N', x', y'). By minimality of u_1, we see that $x', y' \in rg(\xi_1\restriction_{(n+1)\setminus\{x\}})$, and that $\exists$ used case 2 of her amalgamation strategy in round u_1. Hence, (†) $N_{u_1}(x', y') \leq a(i, j)$ for some $i < n-1$ and $j \geq r-1-u_1$.

We claim that $x', y' \in rg(\xi_1(\bar{z}))$. If not, then as $x', y' \in rg(\xi_1\restriction_{(n+1)\setminus\{x\}})$, we have $\xi_1(y) \in \{x', y'\}$. Let $i < n-1$ be such that $\{x', y'\} = \{\xi_1(y), \xi_1(z_i)\}$. We know (by $M \equiv_x N_{u_1}\xi_1$ and figure 6 again) that

$$N_{u_1}(x', y') = N_{u_1}(\xi_1(y), \xi_1(z_i)) = N_{u_1}\xi_1(y, z_i) = M(y, z_i) \leq a(i, j_i).$$

So by (†), $j = j_i$. But by the rest of (†), $j_i \leq r-1-t < r-1-u_1 \leq j$, a contradiction. This proves the claim.

By this claim, the $(n-1)$-tuple $\xi_1(\bar{z})$ was labelled by $\exists$ in round u_1 of the game, using case 2 of her strategy, using the default labelling.

We have now proved that $\exists$ chose the label on $N_{u_1}(\xi_1(\bar{z}))$ in round u_1 using the default labelling. Note that this label is equal to $M(\bar{z})$, since $M \equiv_x N_{u_1}\xi_1$. Pursuing the same argument for N, u_2, ξ_2 shows that $\exists$ also chose the label on the hyperedge $\xi_2(\bar{z})$ in round u_2 of the game. We have $N_{u_2}(\xi_2(\bar{z})) = N(\bar{z})$.

But $M(\bar{z}) = N(\bar{z})$, since $M \equiv_{xy} N$. So $N_{u_1}(\xi_1(\bar{z})) = N_{u_2}(\xi_2(\bar{z}))$. Examination of $\exists$'s default labelling strategy shows that she never chooses the same label for hyperedges in different rounds. Hence, $u_1 = u_2$.

Moreover, she never uses the same label for two hyperedges $\bar{x}, \bar{y}$ in N_{u_1} unless $\bar{x} \sim_{N_{u_1}} \bar{y}$ — i.e., $N_{u_1}(x_i, y_i) = 1'$ for all $i < |\bar{x}| = |\bar{y}|$. Hence, $N_{u_1}(\xi_1(z_i), \xi_2(z_i)) = 1'$ for all $i < n-1$. So, taking the 'u' of case 1 of $\exists$'s strategy to be $u_1 = u_2$, we see that she can use this case in the current round, t. □

5 Ultraproducts of strategies in the game

We have proved that $\exists$ has a winning strategy in the game $G_r^{m,n+1}(\mathfrak{A}(n,r), \Lambda)$, for any infinite set Λ, for each $r < \omega$. We fix $\Lambda = \omega$. Here, we will conclude that she has a winning strategy in $G^{m,n+1}_\omega(\mathcal{B}, {}^\omega\omega/D)$, where $\mathcal{B}$ is any non-principal ultraproduct of the $\mathfrak{A}(n,r)$ over ω and ${}^\omega\omega/D$ is the corresponding non-principal ultrapower of the labelling set ω. Of course, any ultraproduct of atomic relation algebras is also an atomic relation algebra, so the game $G^{m,n+1}_\omega(\mathcal{B}, {}^\omega\omega/D)$ is defined. We can then prove our main results — that $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathsf{S}\mathfrak{Ra}\mathsf{CA}_n$ and that $\mathsf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathsf{S}\mathfrak{Nr}_m\mathsf{CA}_n$ for $3 \leq m \leq n-1$.

Proposition 26 *Let $\mathcal{A}_r$ $(r < \omega)$ be atomic relation algebras, D be a non-principal ultrafilter on ω, and $\mathcal{B}$ be the ultraproduct $\prod_{r<\omega} \mathcal{A}_r/D$. Let $n \geq 5$, $3 \leq m \leq n$. Assume that for each $r < \omega$, $\exists$ has a winning strategy σ_r in the game $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$ for some non-empty set Λ_r. Let $\Lambda \stackrel{\mathrm{def}}{=} \prod_{r<\omega} \Lambda_r/D$. Then she has a winning strategy in the game $G^{m,n}_\omega(\mathcal{B}, \Lambda)$.*

PROOF:

The underlying idea is as follows. $\forall$ and $\exists$ will play $G^{m,n}_\omega(\mathcal{B}, \Lambda)$, and to help her win, $\exists$ will simultaneously play a 'private' game $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$ for each $r < \omega$, using her winning strategy σ_r. At each stage, she will ensure that the play of $G^{m,n}_\omega(\mathcal{B}, \Lambda)$ so far is approximated by a 'large' (i.e., in the ultrafilter) set of private games. In effect, their ultraproduct will be the play of $G^{m,n}_\omega(\mathcal{B}, \Lambda)$.

In more detail, her approach is as follows. Let N be an n-dimensional Λ-hypernetwork over $\mathcal{B}$, let $S \in D$, and for each $r \in S$ let N_r be an n-dimensional $\mathcal{A}_r$-hypernetwork. Let D_S be the ultrafilter $\{S' \in D : S' \subseteq S\}$ on S. As notation, we write $N = \prod_{r \in S} N_r/D$ if:

1. For each $x, y < n$, $N(x,y) = \prod_{r \in S} N_r(x,y)/D_S$ (this is an element of $\mathcal{B}$, up to a natural isomorphism).
2. For every hyperedge $\bar{x} \in {}^{\leq n}n$, $N(\bar{x}) = \prod_{r \in S} N_r(\bar{x})/D_S$. (This is an element of Λ, up to a natural isomorphism.)

In short, $N(\bar{x}) = \prod_{r \in S} N_r(\bar{x})/D_S$ for all $\bar{x} \in {}^{\leq n}n$. Note that if $S' \subseteq S$, $S' \in D$, then also $N = \prod_{r \in S'} N_r/D$.

$\exists$ will privately play a game of $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$ for each $r < \omega$, resulting in a sequence of n-dimensional Λ_r-hypernetworks $N^r_0, N^r_1, \ldots, N^r_t, \ldots$ over $\mathcal{A}_r$, for $t < r$. These private games will proceed in step with the public game $G^{m,n}_\omega(\mathcal{B}, \Lambda)$, which unfolds as a sequence of n-dimensional Λ-hypernetworks N_t $(t < \omega)$ over $\mathcal{B}$. In each round $t < \omega$, we will assume that:

1. $\exists$ has not yet lost $G^{m,n}_\omega(\mathcal{B}, \Lambda)$,
2. there is a set $S_t \in D$, with $s > t$ for all $s \in S_t$, such that $N_u = \prod_{r \in S_t} N^r_u/D$ for each $u < t$, and
3. for each $r \in S_t$, $\exists$ has been using her winning strategy σ_r throughout the play of $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$ so far.

Initially, these conditions are vacuously satisfied. Let $t < \omega$, and assume inductively that they hold just before round t is played. We explain how she will play round t and how she will arrange that the conditions still hold after this round.

There are three cases, depending on which kind of move $\forall$ elects to make in round t of $G^{m,n}_\omega(\mathcal{B}, \Lambda)$. Assume first that he makes an m-dimensional move

(M) by choosing an m-dimensional Λ-hypernetwork M over $\mathcal{B}$. For each edge e of M, $M(e) \in At(\mathcal{B})$. Let $M(e) = (M_r(e) : r < \omega)/D$ for some $M_r(e) \in \mathcal{A}_r$. Similarly, for each hyperedge $\bar{x}$ of M pick representatives $M_r(\bar{x}) \in \Lambda_r$ such that $M(\bar{x}) = (M_r(\bar{x}) : r < \omega)/D$. Thus M_r is a labelled hypergraph, for each $r < \omega$. By Łoś' theorem (see, e.g., [12, theorem 9.5.1]), $S = \{r < \omega : M_r \text{ is a } \Lambda_r\text{-hypernetwork over } \mathcal{A}_r\} \in D$. Let $S_{t+1} = (S \cap S_t) \setminus \{t+1\} \in D$. For each $r \in S_{t+1}$, $\exists$ lets $\forall$ make the m-dimensional move (M_r) in round t of $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$, and continues to use her winning strategy σ_r to respond to this move with a Λ_r-hypernetwork N^r_t with $N^r_t\restriction_m = M_r$. Note that $t < r$, so the game is not over. There is a unique Λ-hypernetwork N_t over $\mathcal{B}$ satisfying $N_t = \prod_{r\in S_{t+1}} N^r_t/D$; Łoś' theorem ensures that N_t is a hypernetwork, and clearly, $N_t\restriction_m = M$. In the game $G^{m,n}_\omega(\mathcal{B}, \Lambda)$, $\exists$ plays N_t in response to $\forall$'s move (M). Conditions 1–3 are preserved.

For the second case, suppose that $\forall$ makes an amalgamation move. Let him play (N_u, N_v, x, y) in $G^{m,n}_\omega(\mathcal{B}, \Lambda)$, where $u, v < t$, $x, y < n$ are distinct, and $N_u \equiv_{xy} N_v$. $\exists$ responds as follows. By the second inductive condition, $N_u = \prod_{r\in S_t} N^r_u/D$, and similarly for N_v. There are finitely many labels in N_u, N_v, so using the definition of equality in the ultraproducts $\mathcal{B}, \Lambda$, it is easily seen that for some $S_{t+1} \subseteq S_t$ with $S_{t+1} \in D$, $t+1 \notin S_{t+1}$, we have $N^r_u \equiv_{xy} N^r_v$ for every $r \in S_{t+1}$. Certainly, every $r \in S_{t+1}$ is bigger than t, so the game $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$ is not over; and (N^r_u, N^r_v, x, y) is a valid $\forall$-move in round t of $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$, for every $r \in S_{t+1}$. $\exists$ lets $\forall$ make this move in each of these private games, and she responds in each of them by continuing with her winning strategy. In $G^{m,n}_r(\mathcal{A}_r, \Lambda_r)$, this yields a hypernetwork N^r_t over $\mathcal{A}_r$ satisfying $N^r_u \equiv_x N^r_t \equiv_y N^r_v$. As before, there is a unique Λ-hypernetwork N_t over $\mathcal{B}$ satisfying $N_t = \prod_{r\in S_{t+1}} N^r_t/D$; Łoś' theorem implies that it is actually a hypernetwork, and also that $N_u \equiv_x N_t \equiv_y N_v$. $\exists$ plays this hypernetwork N_t as her response to $\forall$ in round t of the main game, $G^{m,n}_\omega(\mathcal{B}, \Lambda)$. It is evident that the three conditions above are preserved by this. We have arranged that $t+1 \notin S_{t+1}$, so the second condition is validated.

The third case, where $\forall$ plays a triangle move, is similar and we omit it.

So $\exists$ may proceed in this way forever. She never loses $G^{m,n}_\omega(\mathcal{B}, \Lambda)$ at any stage, and hence she wins it. □

THEOREM 27 *Let $3 \le m < n < \omega$.*

1. *There is no finite set σ of first-order sentences such that for any algebra $\mathcal{C} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$ we have $\mathcal{C} \models \sigma \Leftrightarrow \mathcal{C} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$.*
2. *There is no finite set σ of first-order sentences such that for any relation algebra $\mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ we have $\mathcal{A} \models \sigma \Leftrightarrow \mathcal{A} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$.*

PROOF:

Let us recall and summarise our main results so far. By corollaries 17 and 20, for $r > 0$ we have

$$\mathfrak{C}_r \stackrel{\text{def}}{=} \mathfrak{Ca}(H_m(\mathfrak{A}(n,r),\omega)) \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}, \tag{13}$$

and by theorems 15 and 18,

$$\mathfrak{A}(n,r) \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}. \tag{14}$$

Let D be a non-principal ultrafilter over ω, and define

$$\mathcal{D} = \prod_{r<\omega} \mathfrak{C}_r/D, \qquad \mathcal{B} = \prod_{r<\omega} \mathfrak{A}(n,r)/D. \tag{15}$$

For the first part of the theorem, we aim to show that

$$\mathcal{D} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1} \text{ and } \mathcal{B} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}. \tag{16}$$

Our non-finite axiomatisability results follow from this. To see why, suppose contrarily that σ is a first-order sentence such that for any $\mathcal{C} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$, we have $\mathcal{C} \models \sigma \iff \mathcal{C} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$. Then since $\mathcal{D} = \prod_{r<\omega} \mathfrak{C}_r/D \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ we have $\prod_{r<\omega} \mathfrak{C}_r/D \models \sigma$. Loś' theorem tells us that $\{r < \omega : \mathfrak{C}_r \models \sigma\} \in D$, and D contains infinite sets only. But since $\mathfrak{C}_r \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ for $r > 0$, we deduce that $\mathfrak{C}_r \models \sigma$ for no such r, a contradiction. The non-finite axiomatisability of $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$ over $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ follows similarly.

We now prove (16). We have from theorem 24 and proposition 26 that

$$\exists \text{ has a winning strategy, say } \rho, \text{ in } G^{m,n+1}_\omega(\mathcal{B}, {}^\omega\omega/D). \tag{17}$$

We also know that

$$At(\mathfrak{C}_r) = H_m(\mathfrak{A}(n,r),\omega) \tag{18}$$

for each $r < \omega$, since $\mathfrak{C}_r = \mathfrak{Ca}(H_m(\mathfrak{A}(n,r),\omega))$. From this last equation, standard manipulations of ultraproducts show that $At(\mathcal{D}) = H_m(\mathcal{B}, {}^\omega\omega/D)$.

To prove our results, we seek countable elementary subalgebras $\mathcal{D}' \preceq \mathcal{D}$, $\mathcal{B}' \preceq \mathcal{B}$ and a countable set $\Lambda' \subseteq {}^\omega\omega/D$ such that $At(\mathcal{D}') = H_m(\mathcal{B}', \Lambda')$, and $\exists$ still has a winning strategy in $G^{m,n+1}_\omega(\mathcal{B}', \Lambda')$. Having found these, we will then apply proposition 22.

For this, let Λ_0 be any countable subset of ${}^\omega\omega/D$. Pick any countable elementary subalgebra $\mathcal{B}_0$ of $\mathcal{B}$. Since $\mathcal{B}$ is atomic, $\mathcal{B}_0$ is too and the formula $\phi(x) = \forall y(y < x \leftrightarrow y = 0)$ is satisfied at any atom b of $\mathcal{B}_0$ and so the same formula is satisfied at b in $\mathcal{B}$. Then each atom of $\mathcal{B}_0$ is an atom of $\mathcal{B}$, hence $H_m(\mathcal{B}_0, \Lambda_0) \subseteq H_m(\mathcal{B}, \Lambda_0)$. Let $\mathcal{D}_0$ be a countable elementary subalgebra of $\mathcal{D}$ containing $H_m(\mathcal{B}_0, \Lambda_0)$.

Suppose, for some $i < \omega$, that we have constructed countable elementary subalgebras and countable sets

$$\begin{array}{ll} \mathcal{B}_0 \preceq \cdots \preceq \mathcal{B}_i & \preceq \mathcal{B}, \\ \mathcal{D}_0 \preceq \cdots \preceq \mathcal{D}_i & \preceq \mathcal{D}, \\ \Lambda_0 \subseteq \cdots \subseteq \Lambda_i & \subseteq {}^\omega\omega/D. \end{array}$$

In a play of $G^{m,n+1}_\omega(\mathcal{B}, {}^\omega\omega/D)$, suppose $\forall$ restricts his moves so that for triangle moves he only chooses atoms from $\mathcal{B}_i$ and for m-dimensional moves he only chooses hypernetworks from $H_m(\mathcal{B}_i, \Lambda_i)$. $\exists$'s strategy ρ is winning for this game, and without loss of generality we may assume that ρ is deterministic. Given this restriction on $\forall$'s moves, there are countable sets S_1, L_1 of atoms of $\mathcal{B}$ and labels in ${}^\omega\omega/D$ (respectively) that occur as labels in any hypernetwork that ρ produces in response to a sequence of so-restricted $\forall$-moves. Let S_2 be the set of atoms of $\mathcal{B}$ that occur as the edge label of some hypernetwork in $At(\mathcal{D}_i)$. Let L_2 be the countable subset of ${}^\omega\omega/D$ consisting of all hyperlabels occurring in atoms of $\mathcal{D}_i$.

Let $\Lambda_{i+1} = \Lambda_i \cup L_1 \cup L_2$, and let $\mathcal{B}_{i+1}$ be a countable elementary subalgebra of $\mathcal{B}$ containing $\mathcal{B}_i \cup S_1 \cup S_2$. Now let $\mathcal{D}_{i+1}$ be a countable elementary subalgebra of $\mathcal{D}$ containing $\mathcal{D}_i \cup H_m(\mathcal{B}_{i+1}, \Lambda_{i+1})$. Since $\mathcal{B}_{i+1} \supseteq S_1$ and $\Lambda_{i+1} \supseteq L_1$, if $\forall$'s moves are restricted to using atoms from $\mathcal{B}_i$ and hyperlabels from Λ_i, then $\exists$'s winning strategy ρ produces hypernetworks with edge labels from $\mathcal{B}_{i+1}$ and hyperlabels from Λ_{i+1}: i.e., she plays hypernetworks from $H_{n+1}(\mathcal{B}_{i+1}, \Lambda_{i+1})$. Similarly, since $\mathcal{B}_{i+1} \supseteq S_2$, $\Lambda_{i+1} \supseteq L_2$, by construction of $\mathcal{D}_{i+1}$ we have

$$At(\mathcal{D}_i) \subseteq H_m(\mathcal{B}_{i+1}, \Lambda_{i+1}) \subseteq At(\mathcal{D}_{i+1}). \tag{19}$$

Thus, we get two elementary chains $(\mathcal{B}_i : i < \omega)$ and $(\mathcal{D}_i : i < \omega)$ satisfying (19) for all $i < \omega$. By the elementary chain theorem, $\mathcal{B}' = \bigcup_{i<\omega} \mathcal{B}_i$ is a countable elementary subalgebra of $\mathcal{B}$ and $\mathcal{D}' = \bigcup_{i<\omega} \mathcal{D}_i$ is a countable elementary subalgebra of $\mathcal{D}$. Clearly, $\Lambda' = \bigcup_{i<\omega} \Lambda_i$ is also countable. By construction, ρ gives $\exists$ by restriction a winning strategy in $G^{m,n+1}_\omega(\mathcal{B}', \Lambda')$; and by (19), $At(\mathcal{D}') = \bigcup_{i<\omega} At(\mathcal{D}_i) = \bigcup_{i<\omega} H_m(\mathcal{B}_i, \Lambda_i) = H_m(\mathcal{B}', \Lambda')$.

Since $\exists$ can win the game $G^{m,n+1}_\omega(\mathcal{B}', \Lambda')$, we see by proposition 22 that $\mathcal{B}'$ has an $(n+1)$-dimensional Λ'-hyperbasis $\mathcal{H}$, say, with $\mathcal{H}\restriction_m = H_m(\mathcal{B}', \Lambda')$. By lemma 12, $H_m(\mathcal{B}', \Lambda')$ is an m-dimensional Λ'-hyperbasis for $\mathcal{B}'$, so by proposition 14, $\mathfrak{Ca}(H_m(\mathcal{B}', \Lambda')) \subseteq \mathfrak{Nr}_m(\mathfrak{Ca}(\mathcal{H}))$. But $\mathcal{D}'$ embeds into $\mathfrak{Ca}(H_m(\mathcal{B}', \Lambda'))$ by $d \mapsto \{N \in H_m(\mathcal{B}', \Lambda') : N \leq d\}$. We obtain

$$\mathcal{D}' \subseteq \mathfrak{Ca}(H_m(\mathcal{B}', \Lambda')) \subseteq \mathfrak{Nr}_m(\mathfrak{Ca}(\mathcal{H})). \tag{20}$$

By proposition 8, $\mathfrak{Ca}(\mathcal{H}) \in \mathsf{CA}_{n+1}$, so we have $\mathcal{D}' \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$. Since this class is a variety [21] and $\mathcal{D}' \equiv \mathcal{D}$, it follows that $\mathcal{D} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$, proving the first part of (16).

Similarly, proposition 8 shows that $\mathcal{B}'$ embeds into $\mathfrak{RaCa}(\mathcal{H})$, via $b \mapsto \{N \in \mathcal{H} : N(0,1) \leq b\}$, so $\mathcal{B}' \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. Again, since this class is a variety [8, proposition 52], and $\mathcal{B}' \equiv \mathcal{B}$, we see that $\mathcal{B} \in \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n+1}$. Now (16) is proved, and with it, the theorem. □

COROLLARY 28 *For finite $n \geq 5$, $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$ is not finitely axiomatisable over RA_n or over $\mathsf{RA}_n \cap \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n-1}$.*

PROOF:

It is well-known (see, e.g., [8, proposition 61]) that $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n \subseteq \mathsf{RA}_n$. By remark 25, $\mathfrak{A}(n-1,r) \in \mathsf{RA}_n \setminus \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, so by (14), $\mathfrak{A}(n-1,r) \in (\mathsf{RA}_n \cap \mathbf{S}\mathfrak{Ra}\mathsf{CA}_{n-1}) \setminus \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$, for all $0 < r < \omega$; by (16), an ultraproduct of the $\mathfrak{A}(n-1,r)$ is in $\mathbf{S}\mathfrak{Ra}\mathsf{CA}_n$. The corollary now follows as in theorem 27. □

6 Proof theory

We can easily apply theorem 27 to proof theory with finitely many variables. Because this area is well-known (see, for example, [16, 20, 7, 23, 5]), we do not go into great detail.

In [5, section 7], a Hilbert system for proving m-variable formulas with n-variable proofs (for $m \leq n < \omega$) is given. All formulas are in a fixed countably infinite signature $\mathcal{R}$ consisting of m-ary relation symbols. The logic L_n consists of all such formulas written with variables $v_0, \ldots, v_{n-1}$ only, where all atomic formulas are equalities or of the form $R(v_0, \ldots, v_{m-1})$ for $R \in \mathcal{R}$. That is, the order of variables in atomic formulas other than equalities is fixed. The axioms of $\vdash_{m,n}$ are all propositional tautologies and the following, where $i, j, k < n$ and φ, ψ are L_n-formulas: $\forall v_i(\varphi \to \psi) \to (\forall v_i \varphi \to \forall v_i \psi)$, $\forall v_i \varphi \to \varphi$, $\forall v_i \forall v_j \varphi \to \forall v_j \forall v_i \varphi$, $\forall v_i \varphi \to \forall v_i \forall v_i \varphi$, $\exists v_i \varphi \to \forall v_i \exists v_i \varphi$, $v_i = v_i$, $v_i = v_j \to (v_i = v_k \to v_j = v_k)$, $\exists v_i(v_i = v_i)$, $v_i = v_j \to \forall v_k(v_i = v_j)$ if $k \notin \{i, j\}$, $v_i = v_j \to (\varphi \to \forall v_i(v_i = v_j \to \varphi))$ if $i \neq j$, and $R(v_0, \ldots, v_{m-1}) \to \forall v_i R(v_0, \ldots, v_{m-1})$ if $R \in \mathcal{R}$ and $i \geq m$. The rules of inference are modus ponens and universal generalisation (infer $\forall v_i \varphi$ from φ).

Let Σ be any set of L_n-formulas, and φ an L_m-formula. We now write $\Sigma \vdash_{m,n} \varphi$ if any set S of L_n-formulas containing Σ and all axioms above and closed under the inference rules also contains φ.

Now we identify elements of $\mathcal{R}$ with first-order variables of the language of CA_m. For an L_m-formula φ, define a CA_m-term $\widehat{\varphi}$ by induction: $R(v_0, \ldots, v_{m-1})\widehat{\ } = R$ for $R \in \mathcal{R}$, $\widehat{v_i = v_j} = \mathsf{d}_{ij}$, $\widehat{\neg\varphi} = -\widehat{\varphi}$, $\widehat{\varphi \wedge \psi} = \widehat{\varphi} \cdot \widehat{\psi}$, and $\widehat{\exists v_i \varphi} = \mathsf{c}_i \widehat{\varphi}$. We quote a result from [5, section 7] relating $\vdash_{m,n}$ to neat reducts:

Fact 29 *For any L_m-formula φ, the following are equivalent:*

- $\vdash_{m,n} \varphi$
- $\mathbf{S}\mathfrak{Nr}_m \mathsf{CA}_n \models \widehat{\varphi} = 1$.

For a proof, see [7, 4.3.25]. Note here that the variables (from $\mathcal{R}$) in the equation $\widehat{\varphi} = 1$ are implicitly universally quantified.

Schemata An *m-schema* is simply an L_m-formula $\sigma(R_1, \ldots, R_k)$, where $R_1, \ldots, R_k \in \mathcal{R}$ are the relation symbols occurring in σ. Example: $\forall v_i R(v_0, \ldots, v_{m-1}) \to \forall v_i \forall v_i R(v_0, \ldots, v_{m-1})$, where $i < m$, $R \in \mathcal{R}$. An *m-instance* of σ is a formula of the form $\sigma(\chi_1, \ldots, \chi_r)$, where $\chi_1, \ldots, \chi_r$ are L_m-formulas and each atomic subformula $R_l(v_0, \ldots, v_{m-1})$ of σ has been replaced by χ_l ($l = 1, \ldots, r$). Example: $\forall v_i \varphi \to \forall v_i \forall v_i \varphi$, where φ is any L_m-formula. Note that any m-instance of an m-schema is an L_m-formula.

Theorem 30 *Let $3 \leq m < n < \omega$. There is no finite set of m-schemata whose set Σ of m-instances satisfies*

$$\Sigma \vdash_{m,n} \varphi \iff \vdash_{m,n+1} \varphi$$

for all L_m-formulas φ.

PROOF:

It is enough to prove that there is no single m-schema with the above property. Assume for contradiction that $\sigma(R_1, \ldots, R_k)$ is such a schema. We may assume without loss of generality that σ is a sentence. Let Σ be the set of all m-instances of σ. Clearly, $\sigma \in \Sigma$, so $\Sigma \vdash_{m,n} \sigma$. So $\vdash_{m,n+1} \sigma$, whence by fact 29, $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1} \models \widehat{\sigma} = 1$. By theorem 27, $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$. So there is $\mathcal{A} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \setminus \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ with $\mathcal{A} \models \widehat{\sigma} = 1$. As $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1}$ is a variety, there is an equation, which we can assume is of the form $s = 1$, such that $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_{n+1} \models s = 1$ but $\mathcal{A} \not\models s = 1$. We suppose the variables of the term s come from $\mathcal{R}$; then clearly, there is an L_m-formula φ with $\widehat{\varphi} = s$. So by fact 29, $\vdash_{m,n+1} \varphi$; by assumption, $\Sigma \vdash_{m,n} \varphi$. Hilbert systems are compact, so there is finite $\Sigma_0 \subseteq \Sigma$ with $\Sigma_0 \vdash_{m,n} \varphi$. Let $\varphi' = (\bigwedge \Sigma_0) \to \varphi$, an L_m-formula. As σ is a sentence, it is easily checked that $\vdash_{m,n} \varphi'$. So by fact 29 for n, $\mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n \models \widehat{\varphi'} = 1$. But $\mathcal{A} \in \mathbf{S}\mathfrak{Nr}_m\mathsf{CA}_n$, so $\mathcal{A} \models \widehat{\varphi'} = 1$. That is,

$$\mathcal{A} \models \widehat{\varphi} + \Big(- \prod_{\psi \in \Sigma_0} \widehat{\psi} \Big) = 1. \tag{21}$$

Consider a formula $\psi = \sigma(\chi_0, \ldots, \chi_k) \in \Sigma_0$. Clearly, $\sigma(\chi_0, \ldots, \chi_k)\widehat{}$ is the result of substituting the terms $\widehat{\chi}_1, \ldots, \widehat{\chi}_k$ for the variables $R_1, \ldots, R_k$ in $\widehat{\sigma}$. So $\mathcal{A} \models \widehat{\sigma} = 1$ implies $\mathcal{A} \models \sigma(\chi_0, \ldots, \chi_k)\widehat{} = 1$ (as usual, we implicitly universally quantify $\mathcal{R}$-variables on both sides here). That is, $\mathcal{A} \models \widehat{\psi} = 1$ for every $\psi \in \Sigma_0$. By (21), $\mathcal{A} \models \widehat{\varphi} = 1$. Since we chose s for $\mathcal{A} \not\models s = 1$, this is a contradiction. $\Box$

Problem 2 *As we said in the introduction, [2] showed that $\mathbf{S}\mathfrak{Nr}_n\mathsf{CA}_{n+1}$ is a finitely axiomatisable class. Does theorem 30 hold when $m = n$?*

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