We have included exercises in the text. Some of them are simply there for practising the techniques, and some of them are more significant problems. Here we highlight some of the more important problems whose solution, we think, would make a significant contribution to the research in this field.

1. Prove that **RRA** cannot be axiomatised with a set of first-order sentences using only finitely many variables.

   A related problem is from graph theory. Let $c \geq 2$. Find two finite, non-isomorphic graphs G, H such that $\exists$ has a winning strategy in the c-colour game defined as follows. In round 1, $\forall$ takes one of the c colours and places it on a *set* of nodes of one of the two graphs. $\exists$ takes the same colour and places it on a set of nodes of the other graph. In any subsequent round $\forall$ either takes a colour not already in play and places it on a set of nodes of one of the two graphs, in which case $\exists$ also takes that colour and places it on a set of nodes of the other graph, or he picks up a colour which is already in play ($\exists$ picks up the same colour which is placed on the other graph) and he places that colour on a set of nodes of either graph (and $\exists$ places the same colour on a set of nodes of the other graph).

   At any stage the two graphs have $k \leq c$ colours placed on them; this partitions the nodes of each graph into 2^k corresponding parts. Let us call these parts $G_0, \ldots, G_{2^k-1}$ in G and $H_0, \ldots, H_{2^k-1}$ in H. If for some $i < 2^k$ we have G_i is empty but H_i is not, or the other way round, then $\forall$ wins at this stage. Also if for some $i, j < 2^k$ there is an edge connecting a node in G_i to a node in G_j but there is no edge from a node in H_i to one in H_j (or the other way round) then $\forall$ wins. The play continues for ω rounds. If $\forall$ never wins then $\exists$ wins.

   Essentially, we seek two finite non-isomorphic graphs G, H which cannot be distinguished in a second-order language with c monadic predicate symbols, one binary predicate symbol denoting the edge relation and equality, in which second-order quantification is only allowed over the monadic predicate symbols. If such graphs can be found (say G does not embed into H) then it is possible to use the rainbow construction to construct two relation algebras $\mathcal{A}_{G,H}, \mathcal{A}_{H,H}$ such that the second is representable but the first is not, yet the two algebras cannot be distinguished by any c-variable formula. If this could be done for all finite c then we could solve the previous problem. Currently we have not been able to solve this problem even for $c = 2$.

2. **[Venema]** Find (or show non-existence of) a set of equations $\{e_i : i < \omega\}$ with the following properties:

   - $\mathcal{A} \models \{e_i : i < \omega\} \iff \mathcal{A} \in \mathbf{RRA}$.
   - Each e_i is canonical, i.e. $\mathcal{A} \models e_i \Rightarrow \mathcal{A}^+ \models e_i$.

   Repeat, replacing **RRA** by $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ for $n \geq 5$. Repeat for $\mathbf{RA}_n$ as well.

3. **[Jónsson]** Find all simple relation algebras with no subalgebras other than the whole algebra and the degenerate relation algebra with just one element in its domain.

   This is [5, problem P2]. Maddux stated there that 22 simple relation algebras with no non-trivial subalgebras had been found.

4. If $\mathcal{C} \in \mathbf{RCA}_\omega$ does $\mathcal{C}^+$ necessarily have a complete, ω-dimensional representation?

5. Is the class of relation algebras with a homogeneous representation an elementary class?

6. **[Németi, [6, 2, 1]]** Is the class $\mathbf{IG}_\omega$ (the isomorphism-closure of the ω-dimensional cylindric relativised set algebras in which the unit is closed under substitutions and permutations) a variety, or even a pseudo-elementary class? Is it closed under ultraproducts?

7. [Old problem, stated in [4, p. 681] as well as other sources.] For infinite α, is the equational theory of $\mathbf{D}_\alpha$ decidable?

8. **[partly Maddux]** A relation algebra is said to be *weakly representable* if it has a representation respecting the relation algebra operations ($1', \breve{}, ;$) and $0, 1, \cdot$, but not necessarily $+, -$. The class of weakly representable relation algebras is denoted by **wRRA**. Do we have $\mathbf{wRRA} \subseteq \mathbf{RA}_n$ for some $n \geq 5$? What inclusions hold between **wRRA** and the $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ $(n \geq 5)$?

9. Is **wRRA** a variety? Is it canonical?
10. Is the class **wRRA** of weakly representable relation algebras closed under completions?
11. Are $\mathbf{RA}_5$ and $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_5$ closed under completions?
12. **[Sayed Ahmed]** Which $\mathbf{S}\mathfrak{Nr}_m\mathbf{CA}_n$ for $m < n < \omega$ are closed under completions?
13. **[Németi–Sayed Ahmed]** For finite $n \geq 4$, is $\mathfrak{Ra}\mathbf{CA}_n$ closed under subalgebras? Is it elementary?
14. For which $\alpha \geq 3$ is $\mathrm{Str}\,\mathbf{RCA}_\alpha$ elementary?
15. For finite $n \geq 4$, is every algebra in $\mathbf{S}\mathfrak{Ra}\mathbf{CA}_n$ embeddable in a relation algebra with an n-dimensional cylindric basis?
16. **[Maddux]** Is it decidable whether an arbitrary finite relation algebra has a finite representation?
17. For fixed finite $n \geq 5$, is it decidable whether an arbitrary finite relation algebra has a finite n-dimensional hyperbasis?
18. Let $\vdash_{m,n}$ be the n-variable proof relation of m-variable formulas of [3]. Let $3 \leq m < \omega$. Is there a finite set of m-schemata whose set Σ of m-instances satisfies

$$\Sigma \vdash_{m,m} \phi \iff \vdash_{m,m+1} \phi$$

for all m-variable formulas ϕ?

19. **[Venema]** If a class of structures is closed under ultraproducts, must it be pseudo-elementary?
20. **[5, problems 15, 16]** Is it true that *almost all* finite relation algebras are representable? More precisely, if $RA(n), RRA(n)$ are the numbers of isomorphism types of relation algebras and representable relation algebras (respectively) with no more than n elements, is it the case that

$$\lim_{n\to\infty} \frac{RRA(n)}{RA(n)} = 1?$$

21. **[5, problem 9]** Let $\mathcal{A}$ be a finite relation algebra with a flexible atom. Does $\mathcal{A}$ necessarily have a finite representation?

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