Relation algebras with n-dimensional relational bases
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Abstract

We study relation algebras with n-dimensional relational bases in the sense of Maddux.

Fix n with $3 \leq n \leq \omega$. Write B_n for the class of non-associative algebras with an n-dimensional relational basis, and RA_n for the variety generated by B_n. We define a notion of representation for algebras in RA_n, and use it to give an explicit (hence recursive) equational axiomatisation of RA_n, and to reprove Maddux's result that RA_n is canonical. We show that the algebras in B_n are precisely those that have a complete representation.

Then we prove that whenever $4 \leq n < l \leq \omega$, RA_l is not finitely axiomatisable over RA_n. This confirms a conjecture of Maddux. We also prove that B_n is elementary for $n = 3, 4$ only.

1 Introduction

In this paper we show that the notion of n-dimensional basis provides an 'n-variable' analogue to the classical representation theory for relation algebras. We do this by providing a representation theory for RA_n, the variety generated by all relation algebras (or more strictly, all non-associative algebras) having an n-dimensional relational basis. We prove that an algebra in RA_n has a complete representation exactly when it has an n-dimensional relational basis. Later, we show that the class of all such algebras is elementary for $n = 3, 4$ only. We use a similar proof to confirm a conjecture of Maddux [Mad83]: for every n, l with $4 \leq n < l \leq \omega$, RA_l is not finitely axiomatisable over RA_n.

1.1 Approximations to RRA

The notion of a relation algebra was defined by Jónsson and Tarski ([JT48]; the early history of the subject is surveyed in [Mad91b]), and at the time it was hoped that relation algebras would capture the notion of an algebra of binary relations, in the same way that the axioms for boolean algebra capture the fields of sets (algebras of unary relations). It was soon shown [Lyn50, Mon64] that these hopes were not to be realised. Not every relation algebra is 'representable' (isomorphic to a 'real' algebra of binary relations); worse, the variety RRA of representable relation algebras is not finitely axiomatisable.

Among several reactions to these negative results, one was to seek an explicit infinite axiomatisation of RRA. In [Lyn50] Lyndon laid down a schema generating an infinite sequence

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of first-order axioms which, it was claimed, axiomatised the atomic algebras in RRA. Although this turned out not to be so in general (see [Lyn56]), it is valid for finite relation algebras. These so-called 'Lyndon conditions' can be seen as approximations to RRA.

The Lyndon conditions can be expressed naturally in terms of a two-player game. This was done in [HH97d, HH97b], but much the same ideas were used by Maddux, in [Mad82], for example, and indeed in [Lyn50]. We believe that the explicit use of games concentrates attention on the essential concepts rather than notational details, and so permits more lucid proofs.

The game is played on 'networks'. An (atomic) network is a finite complete directed graph, with each edge labelled by an atom of a fixed atomic relation algebra. The network should satisfy certain consistency conditions: for example, that if a_{xy} is the atom labelling the edge (x, y), then $a_{xy} \leq a_{xz} ; a_{zy}$ for all nodes x, y, z of the network. Crudely, a network can be thought of as a potential fragment of a representation of the algebra as binary relations. However, an arbitrary network may not be a representation, as it may contain 'defects': perhaps there is an edge (x, y) with $a_{xy} \leq b ; c$ for some algebra atoms b, c, yet there is no z such that $a_{xz} = b$ and $a_{zy} = c$. In each round of the game, the first player ('$\forall$', who is male) challenges the second ('$\exists$', who is female) with a defect of this form, and she attempts to cure it by adding a new node z and assigning appropriate labels to the new network edges involving z.

This process brings the network nearer and nearer to being a representation of the algebra. Indeed, for countable simple algebras, if the game is allowed to go on forever, $\forall$ eventually challenges all defects that he ever sees, and $\exists$ never gets stuck, then a genuine representation results. The individual Lyndon conditions essentially[1] express that however $\forall$ plays, $\exists$ can respond without getting stuck for a fixed *finite* number of rounds. The more complex the Lyndon condition, the more rounds $\exists$ can last for.

1.2 Lyndon conditions for finite relation algebras

Suppose the algebra is finite. By a König tree lemma argument, all of the Lyndon conditions taken together imply that $\exists$ can actually survive in the game forever, so that the algebra is representable. And there is a converse: if the given algebra is representable then any finite piece of a representation gives rise to an atomic network (because the algebra is finite), so $\exists$ can use it as a guide in the game and survive any number of rounds.

Thus, we can view $\exists$'s being able to survive r rounds of this game as an rth level approximation to representability, for finite relation algebras.

1.3 Lyndon conditions for arbitrary relation algebras

What about the general case, when the algebra may be infinite? Here, the Lyndon conditions approximate a proper subclass of RRA. First, a representable relation algebra may be atomless, so that the Lyndon conditions do not even make sense in it. But as we will see below, even in the atomic case, while any atomic relation algebra that satisfies the Lyndon conditions is representable, the converse fails.

To see *which* class the Lyndon conditions approximate, we should consider complete representations. Write CRA for the class of all relation algebras that have a *complete representation* — one in which all meets and joins existing in the algebra are preserved. It is easily seen that

[1] For more details see [HH97b].

such an algebra must be atomic. It was shown in [HH97b] that an atomic relation algebra satisfies the Lyndon conditions if and only if it is elementarily equivalent to an algebra that has a complete representation. (Because RRA is elementary, this implies that any atomic algebra satisfying the Lyndon conditions is representable.) Thus, the class that the Lyndon conditions truly approximate is the elementary closure of CRA. This is consistent with the finite case, where we agreed that the Lyndon conditions approximate RRA, because for finite algebras any representation is complete, and elementary equivalence is the same as isomorphism.

Could we say that CRA is a good approximation to RRA in general? On the one hand, yes: CRA is not so far from RRA even for infinite algebras. A theorem of Monk shows that a relation algebra $\mathcal{A}$ is representable if and only if its canonical extension $\mathcal{A}^+$ has a complete representation.[2] This means that RRA is the closure of CRA under taking subalgebras. Hence, because RRA is a variety, it is the variety generated by CRA. We can also observe in passing that RRA is canonical (closed under the map $\mathcal{A} \mapsto \mathcal{A}^+$). Monk's proof is unpublished (his result was reported in [McK66, p. 66]), but several proofs are now known. In [HH97d] a proof by ω-saturation was sketched.

On the other hand, the notion of complete representation is definitely stronger than the original one, even for atomic algebras. CRA itself is not elementary [HH97b]; and since RRA is elementary, it follows that there exist representable relation algebras, and indeed atomic relation algebras satisfying the Lyndon conditions, without a complete representation. Examples of such algebras can be found in [Lyn50, Mad78, HH97b]. There are even algebras where $\exists$ can last forever in the game, but with no complete representation (theorem 50 below). We could say that the Lyndon conditions are 'complete but not sound' over RRA.

In summary, as a series of approximations to RRA the Lyndon conditions are somewhat marred by their ties to complete representations. But this difficulty disappears for finite algebras; and in general, RRA is the variety generated by the class of algebras satisfying the Lyndon conditions.

1.4 n-dimensional relational bases

Another kind of approximation to RRA was introduced by Maddux in [Mad83]. This is the notion of n-dimensional relational basis (where $3 \leq n \leq \omega$). Maddux introduced it in the context of n-variable proof theory, to characterise models in which sequents are valid if and only if they are provable in a certain sequent calculus but using at most n variables in the proofs. Of course, this is a (proof-theoretic) approximation to RRA.

Like the Lyndon conditions, the property of having an n-dimensional basis has a natural interpretation as a game. The game is played on atomic networks over the algebra (only atomic algebras can have n-dimensional bases). It has only two differences from the one described above for the Lyndon conditions: it goes on forever, and, to compensate, the networks are bounded in size by n. This means that if $\forall$ presents $\exists$ with a defect, and in curing it she would have to add a new node which would push the total number of nodes above n, then $\forall$ must first politely remove some node of the existing network to avoid this happening. Thus, we are in the familiar arena of 'n-pebble games' — see, e.g., [Bar77, Imm82, Poi82] for examples of back-and-forth n-pebble games. If $n = \omega$, we recover the game described before.

[2] Recall that $\mathcal{A}^+$ can be taken as the algebra of all sets of ultrafilters of $\mathcal{A}$, with algebraic operations induced from those in $\mathcal{A}$. It embeds $\mathcal{A}$ as a subalgebra and we identify $\mathcal{A}$ with its image under this embedding. For more details see [JT51].

An atomic relation algebra[3] is said to have an *n-dimensional relational basis* if $\exists$ can survive forever in this game, whatever challenges $\forall$ makes. We denote the class of all such algebras by B_n here.

There is no first-order condition on the algebra analogous to a Lyndon condition that expresses that it has an n-dimensional basis, although for finite n there is a simple existential second-order sentence that does. However, there are first-order sentences, which we may call *n-dimensional Lyndon conditions,* expressing that she can last for at least r rounds of the game, for any finite r. So the approximation to RRA of having an n-dimensional basis is itself approximated by the successive n-dimensional Lyndon conditions.

1.5 n-dimensional bases as approximations of RRA

Let us examine the credentials of n-dimensional bases for the role of approximating RRA. If an atomic algebra has an n-dimensional basis for all finite n then it satisfies the Lyndon conditions and so is representable. For *finite* algebras, the converse also holds. So again, having a finite-dimensional relational basis does indeed approximate representability, for finite algebras.

For arbitrary algebras, the successive n-dimensional bases again converge more towards CRA than RRA, as n increases. (The precise situation is described in theorem 50.) However, a solution for this problem is at hand. In [Mad83], Maddux defined MA_n, standing for 'matrix algebra of degree n', to be (in effect) the closure of B_n under taking subalgebras. He renamed it RA_n in [Mad89] and we will use the more recent and suggestive notation here. He showed that RA_n was a variety, the variety generated by B_n, and that $\bigcap_n \mathsf{RA}_n = \mathsf{RRA}$. Hence we may choose to approximate RRA by the RA_n, rather than the B_n.

1.6 n-dimensional bases as analogues of RRA

So B_n and RA_n are respectable classes. It is interesting to treat them, not as approximations to, but as *analogues* of, CRA and RRA.

There is evidence to support such a comparison. We have just seen that RA_n is a variety and is generated by B_n. Moreover, Maddux showed that it is a canonical variety, and indeed that a relation algebra $\mathcal{A}$ is in RA_n if and only if $\mathcal{A}^+ \in \mathsf{B}_n$. These results parallel the case of RRA and CRA exactly. Another similarity is that for *finite* algebras and fixed n, a similar argument to the one for the full Lyndon conditions shows that all the n-dimensional Lyndon conditions together express exactly that the algebra has an n-dimensional basis.

So it seems reasonable to view the classes B_n and RA_n as the n-dimensional analogues of CRA and RRA, respectively. But to decide the matter, we have to provide semantics (a notion of representation) for these algebras.

1.7 Representations of algebras with n-dimensional bases

We prove in this paper that RA_n is the class of all relation algebras that have a certain kind of *relativised representation,* with well-behaved squares or 'cliques' of size n. Relativised representations are the fruit of another reaction to the negative results of Lyndon and Monk mentioned at the start, and now they are a familiar part of algebraic logic. The kind we

[3] Or any atomic semi-associative algebra, and indeed any atomic non-associative algebra [Mad89]; but we leave generalisations to the main part of the paper. Also, this is not Maddux's definition, but it is equivalent: see proposition 31 below.

consider are like those given by Maddux as semantics for the weakly associative relation algebras [Mad82]. The unit (the interpretation of the top element 1 in the representation) is a reflexive, symmetric relation on the base set M of the representation, and all operations, in particular the composition ';', are relativised to the unit. So a pair (x, y) of elements of M will stand in the composite relation $r\,;s$ if there is $z \in M$ with (x, z) standing in the relation r and (z, y) in the relation s, *so long as* (x, y) *already stands in the unit relation.*

But here there is more. Let us call a subset of M a *clique* if all pairs of elements of it lie in the unit. Then we require that for any clique $C \subseteq M$ of size less than n, if $x, y \in C$ and (x, y) lies in the relation $r\,;s$ then there is a clique $C' \supseteq C$ in M containing a point z realising this composition, so that (x, z) lies in the relation r and (z, y) in s. Such a relativised representation is said to be *n-square.* Consideration of the definition of an n-dimensional basis shows that this requirement is very natural here. An n-square representation is 'locally classical', in that without simultaneously considering more than n points, one would never discover using the relation operations that it is relativised. Of course, the definition can be made for any cardinal $n \geq 3$.

We will see in theorem 24 that $\mathsf{RA}_n = \{\mathcal{A} \in \mathsf{NA} : \mathcal{A}$ has an n-square relativised representation$\}$, for $3 \leq n \leq \omega$. Thus, the notion of n-square relativised representation characterises the varieties RA_n.

1.8 Usefulness of relativised representations

As an illustration of the advantage of having a representation theory for the RA_n, we outline a direct argument that shows easily that they converge to RRA. A similar argument appears in [Mad91c, page 112].

Suppose that a simple countable algebra has an n-square relativised representation for every finite n. A compactness argument will show that it has a single relativised representation which is ω-square (n-square for every finite n). A standard step-by-step construction will now build a chain of larger and larger finite cliques in this representation resolving more and more compositional defects, so that their union will have no defects. Being a clique, the union will be a classical representation of the algebra. Since a classical representation is evidently ω-square, we see that $\bigcap_n \mathsf{RA}_n$ and RRA coincide on countable simple algebras. As they are both varieties of relation algebras, it follows by [JT52, theorem 4.15] that they are equal — the result of [Mad83] mentioned above.

There is further profit in having this notion of representation at hand. For example, we can easily use the techniques of [HH97d, HH97a] to give equational axioms for the varieties RA_n $(n < \omega)$. We do this in outline in theorem 35 below. Though the axiomatisation is infinite, and for $n \geq 5$ necessarily so, it is explicit and recursive. By the way, it is decidable, given a finite algebra $\mathcal{A}$, whether it is in RA_n or not. (This was observed in [Mad91b]. A non-deterministic algorithm may simply guess an n^2-ary relation on the atoms of $\mathcal{A}$ and check that it is an n-dimensional basis in the sense of definition 6 below; this algorithm accepts $\mathcal{A}$ if and only if $\mathcal{A} \in \mathsf{B}_n$, if and only if $\mathcal{A}^+ \in \mathsf{B}_n$ (because $\mathcal{A}$ is finite, so that $\mathcal{A} \cong \mathcal{A}^+$), if and only if $\mathcal{A} \in \mathsf{RA}_n$, as required.) It was shown in [HH99c] that the corresponding problem for RRA is undecidable, so this is an advantage of using RA_n instead.

1.9 Complete relativised representations

Now that we have a notion of representation, we can look for a direct n-dimensional analogue of CRA. Earlier, we said that this was B_n, but this class is in a sense not the true n-dimensional analogue of CRA; rather, it is analogous to the class of atomic algebras in which $\exists$ can last forever in the 'Lyndon game' described first. The more natural candidate is the class of all relation algebras that have a *complete* n-square relativised representation. We denote this class by CRA_n here. Note that it could not even be defined without a representation theory.

The analogy between CRA_n and CRA turns out to be very accurate. For example, we will show that an algebra satisfies all the n-dimensional Lyndon conditions if and only if it is elementarily equivalent to an algebra in CRA_n. Further, for $n \geq 5$, CRA_n is not elementary. (But CRA_3 and CRA_4 are elementary!) We can also extend the saturation argument of [HH97d] to prove that $\mathcal{A} \in \mathsf{RA}_n \iff \mathcal{A}^+ \in \mathsf{CRA}_n$, for any relation algebra $\mathcal{A}$. It shows that RA_n is canonical for the same 'reason' as RRA.

It turns out that for $3 \leq n \leq \omega$, we have $\mathsf{CRA}_n = \mathsf{B}_n$. Thus, the above is equivalent to Maddux's result that $\mathcal{A} \in \mathsf{RA}_n \iff \mathcal{A}^+ \in \mathsf{B}_n$.

1.10 Strength of n-dimensional bases

We still have to investigate the relative strength of the n-dimensional-basis approximation to RRA, for varying n. Clearly, $\mathsf{RA}_3 \supseteq \mathsf{RA}_4 \supseteq \cdots \supseteq \mathsf{RA}_\omega$. The inclusions were shown to be strict in [Mad92]. Maddux conjectured [Mad83, page 90] that if $4 \leq n < l \leq \omega$ then RA_l is not finitely based relative to RA_n: that is, there is no first-order sentence σ such that $\mathcal{A} \in \mathsf{RA}_l$ iff $\mathcal{A} \in \mathsf{RA}_n$ and $\mathcal{A} \models \sigma$. This is again similar to the position for RRA, which, as we said, is not finitely axiomatisable [Mon64]. However, there is some evidence on the other side. We already noted that there is an existential second-order sentence expressing that a non-associative algebra is in B_n. For RA_n, recent work of Stebletsova and Venema [SV97] has shown that a non-associative algebra is in RA_n (for $3 \leq n < \omega$) iff it can be embedded in the reduct of a 'Q_n-algebra' to the signature of relation algebras. The Q_n-algebras are the concrete (representable) algebras in a signature permitting the construction of new binary relations from $n \times n$ matrices of old ones, in the manner of Jónsson [Jón91]. For $n \geq 3$, the relation algebra operations are term-definable in this signature. The Q_n-algebras form a finitely axiomatisable canonical variety [SV97, theorem 1]. The conclusion is that RA_n is finitely axiomatisable in an expanded signature.

Nonetheless, in the current paper we confirm Maddux's conjecture:

THEOREM 1 *For each n with $4 \leq n < \omega$, the variety RA_{n+1} is not finitely based relative to RA_n.*

The conjecture follows from this. For supposing that it failed for some $4 \leq n < l < \omega$, let σ be a first-order sentence such that $\mathcal{A} \in \mathsf{RA}_l$ iff $\mathcal{A} \in \mathsf{RA}_n$ and $\mathcal{A} \models \sigma$. Let $m = l - 1 \geq 4$, so that $l = m + 1$. Then because $\mathsf{RA}_n \supseteq \mathsf{RA}_m$, we have $\mathcal{A} \in \mathsf{RA}_{m+1}$ iff $\mathcal{A} \in \mathsf{RA}_m$ and $\mathcal{A} \models \sigma$, contradicting the theorem. The case where $l = \omega$ now follows by first-order compactness, since it cannot be that $\mathsf{RA}_\omega = \bigcap_{n<l<\omega} \mathsf{RA}_l$ is finitely based relative to RA_n if no RA_l $(n < l < \omega)$ is.

In a similar way, we obtain the result discussed earlier:

THEOREM 2 *For any n with $5 \leq n < \omega$, CRA_n is not elementary.*

Our construction in the proofs of these two results originated in [Hir95] and has been used in different forms in [HH97b, Hod97, HH99c]. The version presented here is in a fairly general form (theorem 39) and may be useful in other applications. We will discuss it in more detail later. Games are used throughout, here.

It may be worth repeating here that for $n = 3, 4$, CRA_n is elementary: we have $\mathsf{CRA}_n = \mathsf{B}_n = \{\mathcal{A} \in \mathsf{RA}_n : \mathcal{A} \text{ is atomic}\}$. See corollary 23.

1.11 Summary

Thus, our thesis here, quite consistent with the work of Maddux, is that the notion of n-square relativised representation approximates the classical notion of representation for RRA. But further, being a fully-fledged representation of a kind, it makes RA_n a true analogue of RRA on the n-dimensional level. We also study the 'distance' between the algebras with n-dimensional relational bases, and also their subalgebras, as n increases, proving that RA_{n+1} is not finitely axiomatisable over RA_n for every finite $n \geq 4$.

1.12 Further work

There is yet another approximation to RRA, defined by *neat relation algebra reducts.* Given an n-dimensional cylindric algebra, we obtain such a reduct by deleting all elements that are not cylindrified in dimensions > 1 and defining the relation algebra operations on the remaining elements using the spare dimensions. See, e.g., [HMT85, Mad91a] for details. The class $\mathbf{S}\mathfrak{R}a\mathsf{CA}_n$ consists up to isomorphism of all subalgebras of reducts of n-dimensional cylindric algebras obtained in this way. It can be shown that $\mathbf{S}\mathfrak{R}a\mathsf{CA}_3 \supseteq \mathbf{S}\mathfrak{R}a\mathsf{CA}_4 \supseteq \cdots$, and $\bigcap_{n<\omega} \mathbf{S}\mathfrak{R}a\mathsf{CA}_n = \mathsf{RRA}$, so again we are dealing with a series of approximations to RRA.

In [HH98a] we prove an analogue of theorem 24 for the classes $\mathbf{S}\mathfrak{R}a\mathsf{CA}_n$, and give them a 'hyper-basis' characterisation similar to relational bases. We also recursively axiomatise these classes. It is further shown in [HH98b] that the inclusion $\mathbf{S}\mathfrak{R}a\mathsf{CA}_3 \supset \mathbf{S}\mathfrak{R}a\mathsf{CA}_4 \supset \cdots$, are strict, and in fact that $\mathbf{S}\mathfrak{R}a\mathsf{CA}_{n+1}$ is not finitely axiomatisable over $\mathbf{S}\mathfrak{R}a\mathsf{CA}_n$.

1.13 Plan of paper

Section 2. We recall the definitions and some known results about n-dimensional relational bases.

Section 3. We define the notion of n-square relativised representation formally. We consider complete representations of this form, and we show that for $3 \leq n \leq \omega$, the class of algebras with such a representation is precisely B_n. Then we show that if a non-associative algebra has an n-square relativised representation then its canonical extension has a complete n-square relativised representation. We derive from this that a non-associative algebra has such a representation if and only if it is in RA_n.

Section 4. We define suitable games for analysing the algebras in RA_n more closely. We examine the 'n-dimensional Lyndon conditions', as discussed above, and show how to derive explicit equational axioms for RA_n.

Section 5. In this technical section, we give a construction of non-associative algebras from relational structures, whose game-theoretic properties are closely tied to those of the starting structures.

Section 6. We use the results of section 5 to prove that for $n \geq 5$, RA_n is not finitely axiomatisable over RA_{n-1}, and that B_n is not elementary. We also show that there exist algebras satisfying an 'infinitary' Lyndon condition but having no complete (classical) representation.

Section 7. In the conclusion, we discuss some problems that we came across while preparing the paper and could not answer. They involve completions and the finite base property, for example.

1.14 Notation

Our notation is mostly standard. Algebras of the signature of relation algebras will be denoted $\mathcal{A}, \mathcal{B}$, etc, where $\mathcal{A} = (A, +, -, 0, 1, 1', \breve{}, ;)$. Here, $+$ and $-$ are the boolean sum (join) and complement (negation), 0 and 1 being the bottom and top elements in the boolean order $\leq$, where $x \leq y$ iff $x + y = y$. We write $\cdot$ for boolean meet: $x \cdot y$ abbreviates $-(-x + -y)$. For $r \in A$, $S \subseteq A$, we write $r = \sum S$ if r is the least upper bound in $(A, \leq)$ of the set S. (Note that $\sum S$ may not exist in general.) Similarly, $\prod S$ denotes the greatest lower bound of S in $(A, \leq)$, if it exists. The element $1'$ is the relational identity, $\breve{r}$ denotes the converse of r, and $r \,;s$ the relational composition or product of r and s.

We often use the same notation for a structure or algebra as for its domain: so in the current context we will write $r \in \mathcal{A}$ instead of $r \in A$, etc. This is standard in model theory and algebra. Other notation from model theory, such as r^M for the interpretation of a relation symbol r in a structure M, $\equiv$ for elementary equivalence, and $\preceq$ for elementary substructure, will also be used.

We will often consider *directed graphs*: structures M with a binary relation $E \subseteq M \times M$. Elements of E are called 'edges', and we will often label them by elements of an algebra. A *complete* directed graph is one where $E = M \times M$.

An ordinal is the set of all smaller ordinals. The domain and range of a map θ are denoted by $dom(\theta), rng(\theta)$, respectively.

2 Relational bases

Here, we define n-dimensional relational bases and related ideas formally, and recall some known results on them.

2.1 Maddux's definition of n-dimensional relational basis

Recall that NA is the class of 'non-associative algebras'. These were introduced in [Mad82]. They are the algebras of the signature of relation algebras that satisfy all of the axioms defining relation algebras except perhaps the associativity axiom $\forall xyz(x \,;(y \,; z) = (x \,; y) \,; z)$.

Fact 3 In any non-associative algebra:

1. The operations ; and $\breve{}$ are normal and completely additive [Mad82, theorems 1.10, 1.11].
2. For any x in the algebra, $1' \,; x = x \,; 1' = x$. (This is an axiom of relation algebras.)
3. The map $x \mapsto \breve{x}$ is a boolean algebra automorphism [Mad82, theorem 1.16].
4. $\breve{1} = 1$ and $\breve{1}' = 1'$ [Mad82, theorem 1.13(9,14)].

Let $\mathcal{A}$ be any atomic non-associative algebra. The boolean reduct of $\mathcal{A}$ is a boolean algebra; we write $At\mathcal{A}$ for the set of its atoms (minimal non-zero elements).

Definition 4 An (atomic) network over $\mathcal{A}$, or an (atomic) $\mathcal{A}$-network, is a pair $N = (N_1, N_2)$, where N_1 is a non-empty set of 'nodes', and $N_2 : N_1 \times N_1 \to At\mathcal{A}$ is a 'labelling function', with the following properties:

1. $N_2(x, x) \leq 1'$ for all $x \in N_1$
2. $N_2(x, y)^{\smile} = N_2(y, x)$ for all $x, y \in N_1$
3. $N_2(x, y) \leq N_2(x, z) ; N_2(z, y)$ for all $x, y, z \in N_1$.

If $N_2(x, y) \leq 1' \Leftrightarrow (x = y)$ for all $x, y \in N_1$ then we say that the network is *strict*.

If $N = (N_1, N_2), N' = (N'_1, N'_2)$ are networks, we say that N is a subnetwork of N', that N' extends N, and write $N \subseteq N'$, if $N_1 \subseteq N'_1$ and $N_2(x, y) = N'_2(x, y)$ for all $x, y \in N_1$.

One may think of an atomic network N as a complete labelled directed graph on the nodes N_1, the label on the edge (x, y) being $N_2(x, y)$.

Conventions. In this paper, with the exception of section 4.3, all networks will be atomic, so we will not keep saying so.

To free up some suffixes, and in line with our identifying (notationally) a structure with its domain, we generally write N for any of N, N_1, N_2, distinguishing them by context. We will write $|N|$ for $|N_1|$, i.e., the number of nodes of the network N. In case of need, we will refer to the set of nodes of N as $dom(N)$, the *domain* of N.

Definition 5 If N, N' are networks with the same domain, and $x \in N$, we write $N \equiv_x N'$ if $N(y, z) = N'(y, z)$ for all $y, z \in N \setminus \{x\}$.

Now let α be an ordinal, with $3 \leq \alpha \leq \omega$.

Definition 6 (Maddux, 1983, §2) Let $\mathcal{A}$ be an atomic non-associative algebra,[4] and $\mathcal{M}$ be a set of $\mathcal{A}$-networks with domain α. We say that $\mathcal{M}$ is an *α-dimensional (relational) basis* for $\mathcal{A}$ if:

1. If $a \in At\mathcal{A}$ then there is $N \in \mathcal{M}$ with $N(0, 1) = a$.
2. If $N \in \mathcal{M}$, $i, j, k < \alpha$ with $k \neq i, j$, $a, b \in At\mathcal{A}$, and $N(i, j) \leq a ; b$, then there is $N' \in \mathcal{M}$ with $N' \equiv_k N$, $N'(i, k) = a$, $N'(k, j) = b$.

Definition 7 We write B_α for the class of all atomic non-associative algebras $\mathcal{A}$ such that there exists an α-dimensional basis for $\mathcal{A}$.

Definition 8 (Maddux, 1983, §4) An algebra $\mathcal{A}$ of the similarity type of non-associative algebras is said to be a relation algebra of dimension α, in symbols $\mathcal{A} \in \mathsf{RA}_\alpha$, if there exists a complete atomic non-associative algebra which has an α-dimensional basis and contains $\mathcal{A}$ as a subalgebra.

[4]In [Mad83], Maddux defines relational bases for semi-associative algebras, but the definition is generalised to non-associative algebras in [Mad89].

2.2 Facts

FACT 9 We now quote some useful facts about these classes, mostly taken from [Mad83].

1. For $3 \leq \alpha \leq \omega$, $\mathsf{RA}_\alpha = \mathsf{S}(\mathsf{B}_\alpha)$, the class of subalgebras of algebras in B_α.

   '$\subseteq$' is trivial. To see '$\supseteq$', suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{B} \in \mathsf{B}_\alpha$. So $\mathcal{B}$ is atomic and has an α-dimensional basis. Let $\mathcal{C}$ be the completion of $\mathcal{B}$ — equivalently, the full complex algebra over the atom structure[5] of $\mathcal{B}$. As $\mathcal{C}$ has the same atom structure as $\mathcal{B}$, any α-dimensional basis for $\mathcal{B}$ is also one for $\mathcal{C}$. So $\mathcal{C}$ also has an α-dimensional basis. But $\mathcal{C}$ is complete and atomic, and (by [Mad82, theorem 3.13(3,1)]) a non-associative algebra that contains $\mathcal{A}$ as a subalgebra. We obtain $\mathcal{A} \in \mathsf{RA}_\alpha$.

2. If $3 \leq k \leq l \leq \omega$ then $\mathsf{RA}_k \supseteq \mathsf{RA}_l$ [Mad83, theorem 3].

3. $\mathsf{SA} = \mathsf{RA}_3$ [Mad91c, theorem 35]. Here, SA is the class of all semi-associative algebras.

4. $\mathsf{RA} = \mathsf{RA}_4$ [Mad83, theorem 6(2)]. RA is the class of all relation algebras.

5. For $n = 3, 4$, every atomic algebra $\mathcal{A} \in \mathsf{RA}_n$ has an n-dimensional basis (e.g., the set of *all* atomic $\mathcal{A}$-networks with domain n) and is therefore in B_n [Mad83, theorems 4 and 5]. Hence, for $n \in \{3, 4\}$, $\mathsf{B}_n = \{\mathcal{A} \in \mathsf{RA}_n : \mathcal{A} \text{ is atomic}\}$.

6. $\mathsf{RRA} = \mathsf{RA}_\omega = \bigcap_{3 \leq n < \omega} \mathsf{RA}_n$ [Mad83, theorems 6(3), 10].

7. RA_α is a variety for $3 \leq \alpha \leq \omega$ [Mad83, theorem 9].

8. RA_α is closed under taking canonical (or embedding) algebras, for $3 \leq \alpha \leq \omega$; indeed, if $\mathcal{A} \in \mathsf{RA}_\alpha$ then its canonical extension $\mathcal{A}^+$ is in B_α [Mad83, theorem 8]. See corollary 27 below for another proof, and [JT51] for details of canonical extensions.

9. Hence, for finite $\mathcal{A}$, we have $\mathcal{A} \in \mathsf{B}_\alpha$ iff $\mathcal{A} \in \mathsf{RA}_\alpha$ (because if $\mathcal{A}$ is finite then $\mathcal{A}^+ = \mathcal{A}$).

3 Representation theory

The aim now is to develop a representation theory for algebras associated with n-dimensional relational bases. In [Mad82] it was shown that the weakly associative relation algebras are precisely those that have relativised representations in which the unit is a reflexive and symmetric relation. This can be generalised to the algebras in RA_n: the unit remains reflexive and symmetric, but it has further properties.

Notation Until the end of this section we fix n with $3 \leq n \leq \omega$, and we let κ denote any cardinal (with $\kappa \geq 3$).

Let $\mathcal{A}$ be a non-associative algebra. Let us regard $\mathcal{A}$ as a binary relational signature (or similarity type). That is, each element of (the domain of) the algebra $\mathcal{A}$ will be regarded as a binary relation symbol. (This will not lead to ambiguity: for $r \in \mathcal{A}$, if we write $r(x, y)$, we are thinking of r as a relation symbol, but if we write simply r, we are thinking of r as an element of $\mathcal{A}$.) We define some first-order theories in this signature, and then, our notion of representation.

[5] For a full definition of an atom structure, see [Mad82].

Definition 10

1. $T_{\mathcal{A}}$ is the theory consisting of the following axioms:

$$\begin{array}{rl}
\forall xy[1'(x,y) \leftrightarrow (x=y)] & \\
\forall xy[r(x,y) \leftrightarrow s(x,y) \vee t(x,y)] & \text{for each } r,s,t \in \mathcal{A} \text{ with } \mathcal{A} \models r = s+t \\
\forall xy[1(x,y) \rightarrow (r(x,y) \leftrightarrow \neg s(x,y))] & \text{for each } r,s \in \mathcal{A} \text{ with } \mathcal{A} \models r = -s \\
\forall xy[r(x,y) \leftrightarrow s(y,x)] & \text{for each } r,s \in \mathcal{A} \text{ with } \mathcal{A} \models r = \breve{s} \\
\forall xy[1(x,y) \wedge \exists z(s(x,z) \wedge t(z,y)) \rightarrow r(x,y)] & \text{for each } r,s,t \in \mathcal{A} \text{ with } \mathcal{A} \models r = s\,;t \\
\exists xy\, r(x,y) & \text{for each } r \in \mathcal{A} \text{ with } \mathcal{A} \models r \neq 0.
\end{array}$$

2. For $3 \leq k < \omega$, let the theory $T^k_{\mathcal{A}}$ consist of all sentences in $T_{\mathcal{A}}$, together with axioms of the form

$$\forall x_0,\dots,x_{k-2}\Big(r(x_i,x_j) \wedge \bigwedge_{l,m<k-1} 1(x_l,x_m)$$
$$\rightarrow \exists x_{k-1}(s(x_i,x_{k-1}) \wedge t(x_{k-1},x_j) \wedge \bigwedge_{l,m<k} 1(x_l,x_m))\Big)$$

   for each $r,s,t \in \mathcal{A}$ with $\mathcal{A} \models r = s\,;t$ and each $i,j < k-1$.

3. Define $T^\omega_{\mathcal{A}} = \bigcup_{3\leq k<\omega} T^k_{\mathcal{A}}$.

Definition 11 A *κ-square relativised representation* of $\mathcal{A}$ is a model M of $T_{\mathcal{A}}$ with the property that for any subset $C \subseteq M$ with $|C| < \kappa$ and $M \models 1(c,d)$ for all $c,d \in C$, and for any $x,y \in C$ and $r,s \in \mathcal{A}$ with $M \models [r\,;s](x,y)$, there is $z \in M$ with $M \models r(x,z) \wedge s(z,y)$ and $M \models 1(c,d)$ for all $c,d \in C \cup \{z\}$.

It is clear from the definitions that an $\mathcal{A}$-structure M is an n-square relativised representation of $\mathcal{A}$ if and only if $M \models T^n_{\mathcal{A}}$. (Here, $3 \leq n \leq \omega$ as usual.) One can also define these relativised representations using the conventional algebraic approach by homomorphisms; we use first-order theories instead because they will be needed in lemma 26 below.

It will be useful to prove the following lemma immediately.

Lemma 12 *Let M be any κ-square relativised representation of the non-associative algebra $\mathcal{A}$. Then as boolean algebras, we have*

$$(\mathcal{A},+,-,0,1) \cong (\{r^M : r \in \mathcal{A}\},\cup,\setminus,\emptyset,1^M),$$

where, for $r \in \mathcal{A}$, r^M denotes the interpretation of the binary relation symbol r as a binary relation on M (so $r^M \subseteq M^2$). Also, 1^M is a reflexive and symmetric relation on M.

PROOF:

The isomorphism is evidently $r \mapsto r^M$. By the axioms for $+$ and $-$ in $T_{\mathcal{A}}$, $(r+s)^M = r^M \cup s^M$ and $(-r)^M = 1^M \setminus r^M$. Obviously, $1 \mapsto 1^M$. We check that 0 maps to $\emptyset$. If $M \models 0(x,y)$ then by the $+$-axiom, $M \models 1(x,y)$, so by the axiom for $-$, $M \models \neg 0(x,y)$, a contradiction. So $0^M = \emptyset$. Finally, we check that the map $r \mapsto r^M$ is one-to-one. We have just seen that this map is a boolean homomorphism, so it is enough to show that $r^M \neq \emptyset$ if $r \neq 0$ in $\mathcal{A}$. But there is an axiom of $T_{\mathcal{A}}$ saying exactly this.

If $x \in M$ then $M \models 1'(x,x)$ by the identity axiom; and as $1' \leq 1$, we have $M \models 1(x,x)$ by the above. So 1^M is reflexive. For symmetry, note that in non-associative algebras we have $\breve{1} = 1$ (see fact 3(4)); now use the axiom for converse in $T_{\mathcal{A}}$. $\Box$

We will prove that the non-associative algebras having an n-square relativised representation (for $3 \leq n \leq \omega$) are precisely those in RA_n. But first we deal with the simpler case of the algebras in B_n. The general case will follow quite easily by a saturation argument.

3.1 Complete n-square relativised representations

Here, we will show that the non-associative algebras in B_n are precisely those having a 'complete' n-square relativised representation. The cardinals n, κ remain fixed ($3 \leq n \leq \omega$).

Definition 13 A κ-square relativised representation M of an algebra $\mathcal{A}$ (for $\kappa \geq 3$) is said to be *complete* if whenever $r \in \mathcal{A}$, $S \subseteq \mathcal{A}$, and $r = \sum S$ in $\mathcal{A}$, then $M \models \forall xy(r(x,y) \leftrightarrow \bigvee_{s \in S} s(x,y))$, the possibly infinitary disjunction here being given semantics in the usual (classical) way.

We write CRA_κ for the class of all non-associative algebras that have a complete κ-square relativised representation.

Thus, we aim to show that $\mathsf{CRA}_n = \mathsf{B}_n$ for each n. The reader should beware: CRA_κ, CRA_n are defined by representations, while RA_n is not. We will see in theorem 24 that RA_n can be equivalently defined in terms of representations, so this notational discrepancy is only temporary.

Remark 14 Any representation of a finite algebra is complete. By the de Morgan laws and lemma 12, we see that a κ-square relativised representation M is complete iff whenever $r \in \mathcal{A}$, $S \subseteq \mathcal{A}$, and $r = \prod S$ in $\mathcal{A}$, then $M \models \forall xy(r(x,y) \leftrightarrow \bigwedge_{s \in S} s(x,y))$.

We begin with a more 'concrete' description of complete representations.

Definition 15 A κ-square relativised representation M of an algebra $\mathcal{A}$ is said to be *atomic* if for every $x, y \in M$ with $M \models 1(x,y)$, there is an atom $a \in \mathcal{A}$ with $M \models a(x,y)$.

Of course, by lemma 12, for each x, y the atom a in the definition will be unique.

Lemma 16 *Let $\mathcal{A}$ be any non-associative algebra. A κ-square relativised representation M of $\mathcal{A}$ is complete if and only if it is atomic.*

PROOF:

Let $M \models T_{\mathcal{A}}$ (κ-squareness is not needed here). We use lemma 12 freely in the proof.

First suppose that M is an atomic relativised representation. We must prove that it is complete. Let $r = \sum S$ in $\mathcal{A}$. We must show that $M \models \forall xy(r(x,y) \leftrightarrow \bigvee_{s \in S} s(x,y))$. The implication '$\leftarrow$' is clear. For '$\rightarrow$', suppose that $x, y \in M$ and $M \models r(x,y)$. As M is atomic, there is $a \in At\mathcal{A}$ such that $M \models a(x,y)$. Then $r \cdot a \neq 0$. As a is an atom, $a \leq r$ in $\mathcal{A}$, whence $a \leq s$ for some $s \in S$

(otherwise, $r \cdot -a$ would be a smaller upper bound for S, contradicting $r = \sum S$). So $M \models s(x,y)$, as required.

Now suppose that M is a complete relativised representation. Let $x, y \in M$ be such that $M \models 1(x,y)$. We seek an atom $a \in A$ with $M \models a(x,y)$. Let $S = \{s \in \mathcal{A} : M \models s(x,y)\}$. S is an ultrafilter on $\mathcal{A}$. If it is principal, we are done, as it will contain an atom. If not, we have $\prod S = 0$. But then since M is complete, we have $M \models \forall xy(0(x,y) \leftrightarrow \bigwedge_{s \in S} s(x,y))$, so $M \models 0(x,y)$, which is impossible. □

PROPOSITION 17 *Let $\mathcal{A} \in \mathsf{CRA}_n$. Then $\mathcal{A}$ is atomic, and $\mathcal{A} \in \mathsf{B}_n$.*

PROOF:

Take non-zero $r \in \mathcal{A}$. We show that there is an atom beneath it. Let $M \models T^n_{\mathcal{A}}$ be a complete n-square relativised representation of $\mathcal{A}$. Again, we use lemma 12 freely on M. Pick $x, y \in M$ with $M \models r(x,y)$. By lemma 16, there is an atom $a \in \mathcal{A}$ with $M \models a(x,y)$. So $a \cdot r \neq 0$, and as a is an atom, $a \leq r$.

For the second part, suppose that $M \models T^n_{\mathcal{A}}$ is a complete (and hence by lemma 16, atomic) relativised representation of $\mathcal{A}$. We will extract from M an n-dimensional basis for $\mathcal{A}$. Let Φ be the set of all maps $f : n \to M$ such that (i) $M \models 1(f(i), f(j))$ for all $i, j < n$, and (ii) $rng(f)$ is finite (of course this restriction has no effect if $n < \omega$). For $f \in \Phi$, let $\nu_f : n \times n \to At\mathcal{A}$ be given by: $\nu_f(i,j)$ is the unique atom $a \in \mathcal{A}$ such that $M \models a(f(i), f(j))$, for each $i, j < n$; such an atom exists because M is atomic and $M \models 1(f(i), f(j))$. Finally, define $\mathcal{M} = \{(n, \nu_f) : f \in \Phi\}$. It is a simple matter to check that $\mathcal{M}$ is an n-dimensional relational basis for $\mathcal{A}$. □

We remark that this proof generalises to arbitrary cardinals κ without change (see [Mad91c, definition 34] for the generalisation of α-dimensional relational bases to arbitrary ordinals α).

To prove the converse of proposition 17, we extract a complete n-square relativised representation of an atomic non-associative algebra $\mathcal{A}$ directly from an n-dimensional basis for $\mathcal{A}$. This essentially involves treating an n-dimensional basis $\mathcal{M}$ as a *saturated set of mosaics* in the sense of Németi [Ném86, Ném95], and constructing a representation from it by resolving all 'defects'.

PROPOSITION 18 *Let $\mathcal{A} \in \mathsf{B}_n$. Then $\mathcal{A} \in \mathsf{CRA}_n$.*

PROOF:

Take $\mathcal{A} \in \mathsf{B}_n$, and let $\mathcal{M}$ be an n-dimensional basis for $\mathcal{A}$. We will build a chain of possibly uncountable labelled, directed graphs M_h $(h < \omega)$. Their union, M_ω, will be the representation we seek.

To define M_0, we need to identify points in networks in $\mathcal{M}$ if they are related by an identity atom.

DEFINITION 19 A binary relation $\sim$ on (the domain of) an $\mathcal{A}$-network N is said to be a *congruence* if it is an equivalence relation and for all nodes $i, i', j, j' \in N$, if $i \sim i'$ and $j \sim j'$ then $N(i,j) = N(i',j')$.

If $\sim$ is a congruence on N, define an $\mathcal{A}$-network $N/\sim$ by: $[N/\sim](u,v) = N(i,j)$ for any $i \in u, j \in v$, where u, v are any $\sim$-classes in N. (Clearly, this is well-defined.)

It is easy to check that $N/{\sim}$ is an $\mathcal{A}$-network. Suppose $u, v, w \in N/{\sim}$, and take $i \in u$, $j \in v$, $k \in w$. Then $[N/{\sim}](u,u) = N(i,i) \leq 1'$, and $[N/{\sim}](u,v) = N(i,j) = N(j,i)^{\smile} = [N/{\sim}](v,u)^{\smile}$ For composition, we have $[N/{\sim}](u,v) = N(i,j) \leq N(i,k)\,;N(k,j) = [N/{\sim}](u,w)\,;[N/{\sim}](w,v)$, as required.

For $N \in \mathcal{M}$, define a binary relation $\sim$ on $n = dom(N)$ by: $i \sim j$ iff $N(i,j) \leq 1'$. Of course, $\sim$ depends on N.

LEMMA 20 *$\sim$ is a congruence on N. Moreover, $N/{\sim}$ is a strict $\mathcal{A}$-network.*

PROOF:

First we check that $\sim$ is an equivalence relation. Reflexivity of $\sim$ follows from the definition of network. Symmetry holds because in non-associative algebras, the map $x \mapsto \breve{x}$ is a boolean automorphism fixing $1'$ (fact 3(3,4)): so we have $N(i,j) \leq 1' \Rightarrow N(j,i) = N(i,j)^{\smile} \leq \breve{1}' = 1'$. Transitivity holds because the product ';' is additive in $\mathcal{A}$ (fact 3(1)), and $1'\,;1' = 1'$ (fact 3(2)): so if $i \sim j \sim k$ then $N(i,k) \leq N(i,j)\,;N(j,k) \leq 1'\,;1' = 1'$, whence $i \sim k$.

Now we let $i \sim i'$, $j \sim j'$, and check that $N(i,j) = N(i',j')$. Since $1'\,;a = a\,;1' = a$ for any atom $a \in \mathcal{A}$, and ';' is additive, we have

$$\begin{aligned} N(i,j) &\leq N(i,i')\,;N(i',j) \\ &\leq N(i,i')\,;(N(i',j')\,;N(j',j)) \\ &\leq 1'\,;(N(i',j')\,;1') \\ &= N(i',j'). \end{aligned}$$

As $N(i,j)$ and $N(i',j')$ are atoms, they are equal. So $\sim$ is a congruence, as required.

Now, for strictness, if $[N/{\sim}](u,v) \leq 1'$ then $i \sim j$ for any $i \in u$, $j \in v$, so that $u = v$. □

We are now going to define the chain M_h ($h < \omega$). Recall that the M_h are directed labelled graphs; they will not be complete graphs. We will require (inductively) that for each $h < \omega$, M_h satisfies conditions 1–4 below:

1. The set of edges forms a reflexive and symmetric binary relation on M_h.
2. Each directed edge (x,y) of M_h is labelled by a unique atom of $\mathcal{A}$, written $M_h(x,y)$.
3. For all $x, y \in M_h$, (i) $M_h(x,x) \leq 1'$, and (ii) if (x,y) is an edge of M_h and $M_h(x,y) \leq 1'$ then $x = y$.

For such a graph M_h, and a network $N \in \mathcal{M}$, a map $\nu : N \to M_h$ is said to be an *embedding* if whenever $i, j < n$ then $(\nu(i), \nu(j))$ is an edge of M_h and $M_h(\nu(i), \nu(j)) = N(i,j)$. Note that embeddings need not be one-to-one, but they do preserve atoms under $1'$, an adequate substitute. Say (as usual in graph theory) that a subset C of M_h is a *clique* if the induced subgraph of M_h with domain C is a complete directed graph.

We further require of M_h:

4. Any clique in M_h is contained in $rng(\nu)$ for some $N \in \mathcal{M}$ and some embedding $\nu : N \to M_h$.

Now every $N/\sim$, for $N \in \mathcal{M}$, can be seen as a complete labelled directed graph in the obvious way; and in such a view, by lemma 20, the properties 1–3 above hold. Define M_0 to be the disjoint union $\dot{\bigcup}_{N\in\mathcal{M}} N/\sim$, the edges and labels of M_0 being those of the individual $N/\sim$. If $N \in \mathcal{M}$, the map $\nu : N \to M_0$ taking each $i \in N$ to its $\sim$-class in $N/\sim$ is an embedding. Using this, it is clear that M_0 meets the requirements 1–4 above.

Assume inductively that M_h is defined for some $h < \omega$. Then we define the extension M_{h+1} of M_h so that for every quadruple (N, ν, k, N'), where $N, N' \in \mathcal{M}$, $k < n$, $N \equiv_k N'$, and $\nu : N \to M_h$ is an embedding, the restriction ν_k of ν to $n \setminus \{k\}$ extends to an embedding $\nu' : N' \to M_{h+1}$. We do this as follows.

- If $N'(k, i) \leq 1'$ for some $i \neq k$, then we may (must) set $\nu'(k) = \nu(i)$. This is well-defined if there are several such i, and is an embedding. No change to M_h is made for these (N, ν, k, N').
- For each other (N, ν, k, N'), we adjoin a new point $\pi = \pi_{(h,N,\nu,k,N')}$ to M_h, and add just the following edges to M_h: (π, π), and $(\pi, \nu(i)), (\nu(i), \pi)$ for each $i \in n \setminus \{k\}$.
- The new edges are labelled by atoms as follows: (π, π) is labelled by $N'(k, k)$, $(\pi, \nu(i))$ by $N'(k, i)$, and $(\nu(i), \pi)$ by $N'(i, k)$.
- We may extend ν_k to ν' defined on k, by setting $\nu'(k) = \pi$. Because $N' \equiv_k N$, this is an embedding $: N' \to M_{h+1}$.
- For distinct (N, ν, k, N'), the new points $\pi_{(h,N,\nu,k,N')}$ are distinct.
- M_{h+1} will consist of M_h, with its old labels and edges, together with all these new points, edges, and labels.

It is easy to check that the properties 1–4 above are preserved by these actions. For 4, note that any clique in M_{h+1} is either a clique in M_h, for which we have the result inductively, or else it contains a new point $\pi_{(h,N,\nu,k,N')}$, in which case it is contained in $rng(\nu')$, since the only edges involving $\pi_{(h,N,\nu,k,N')}$ lie in this set.

Now let $M = M_\omega = \bigcup_{h<\omega} M_h$. Define M as an $\mathcal{A}$-structure by:

$$M \models r(x, y) \iff \exists h < \omega(x, y \in M_h \wedge (x, y) \text{ is an edge of } M_h \wedge M_h(x, y) \leq r),$$

for each $r \in \mathcal{A}$ and $x, y \in M$.

Lemma 21 *M is an n-square relativised representation of $\mathcal{A}$.*

PROOF:

To see that $M \models \forall xy(1'(x, y) \leftrightarrow x = y)$, use property 3 above. The boolean clauses are easy to check.

The axiom for converse, $\forall xy(r(x, y) \leftrightarrow \breve{r}(y, x))$, holds in M because of properties 1 and 4. For if $M \models r(x, y)$, there is $h < \omega$ with (x, y) an edge of M_h and $M_h(x, y) \leq r$. By the cited properties, there are $N \in \mathcal{M}$, an embedding $\nu : N \to M_h$, and $i, j < n$ with $\nu(i) = x$ and $\nu(j) = y$. Then $N(i, j) = M_h(x, y) \leq r$, and as N is a network and the map $x \mapsto \breve{x}$ preserves $\leq$, we have $M_h(y, x) = N(j, i) \leq \breve{r}$. So $M \models \breve{r}(y, x)$, as required.

The axiom $\exists xy\, r(x,y)$ for $r \neq 0$ in $\mathcal{A}$ holds because for any atom $a \leq r$ there is $N \in \mathcal{M}$ with $N(0,1) = a$. This is because $\mathcal{M}$ is an n-dimensional basis for $\mathcal{A}$. Since $N/{\sim} \subseteq M_0$, there are $x, y \in M_0$ with $M_0(x,y) = a$. Hence, $M \models r(x,y)$.

Now consider the product axioms. First, suppose that $x, y, z \in M$ and that $M \models 1(x,y) \wedge r(x,z) \wedge s(z,y)$. We require $M \models [r\,;s](x,y)$. Clearly, there are $h < \omega$ with $x, y, z \in M_h$, and atoms a, b, c with $b \leq r$, $c \leq s$, and $M_h(x,y) = a$, $M_h(x,z) = b$, $M_h(z,y) = c$. By property 4, there are $N \in \mathcal{M}$, an embedding $\nu : N \to M_h$, and $i, j, k < n$ with $\nu(i) = x$, $\nu(j) = y$, $\nu(k) = z$. By network rules, $a = N(i,j) \leq N(i,k)\,; N(k,j) = b\,; c$. By additivity of the product in $\mathcal{A}$, we have $b\,; c \leq r\,; s$. So by definition, $M \models [r\,; s](x,y)$.

Finally, let $C \subseteq M$ with $|C| < n$ and $M \models 1(c,d)$ for all $c, d \in C$. Let $x, y \in C$ and $r, s \in \mathcal{A}$ and suppose that $M \models [r\,; s](x,y)$. We must find $z \in M$ with $M \models r(x,z) \wedge s(z,y)$ and $M \models 1(c,d)$ for all $c, d \in C \cup \{z\}$.

As $|C| < \omega$, there is $h < \omega$ with $C \subseteq M_h$. Evidently, C is a clique in M_h. By property 4, there is $N \in \mathcal{M}$ and an embedding $\nu : N \to M_h$ with $C \subseteq rng(\nu)$. For each $c \in C$ choose $c' < n$ with $\nu(c') = c$. By the complete additivity of the product in $\mathcal{A}$ (fact 3(1)), we may choose atoms $a, b \in \mathcal{A}$ with $a \leq r, b \leq s$, and $a\,; b \geq M_h(x,y)$. We may also choose $k < n$ not of the form c' for any $c \in C$; k exists because $|C| < n$. As $\mathcal{M}$ is an n-dimensional relational basis, there is $N' \in \mathcal{M}$ with $N' \equiv_k N$, $N'(x',k) = a$, and $N'(k,y') = b$. By definition of M_{h+1}, the restriction of ν to $n \setminus \{k\}$ extends to an embedding $\nu' : N' \to M_{h+1}$. Let $z = \nu'(k)$. Then $M_{h+1}(x,z) = N'(x',k) = a$, and $M_{h+1}(z,y) = N'(k,y') = b$. So by choice of a, b, and the definition of M, we have $M \models r(x,z) \wedge s(z,y)$. Also, by definition of M_{h+1}, $rng(\nu')$ is a clique in M_{h+1} containing C, z, and it follows that $M \models 1(c,d)$ for all $c, d \in C \cup \{z\}$. $\Box$

Now by definition of M, if $M \models 1(x,y)$ then there is an atom a of $\mathcal{A}$ with $M \models a(x,y)$. Hence, by lemma 16, M is a complete n-square relativised representation of $\mathcal{A}$. $\Box$

By propositions 17 and 18, we conclude:

THEOREM 22 *For any n with $3 \leq n \leq \omega$, $\mathsf{CRA}_n = \mathsf{B}_n \subseteq \mathsf{RA}_n$.*

Any non-atomic $\mathcal{A} \in \mathsf{RA}_n$ witnesses $\mathsf{RA}_n \neq \mathsf{B}_n$. Corollary 27 below shows that restricting to finite algebras, we have $\mathsf{CRA}_n = \mathsf{B}_n = \mathsf{RA}_n$ — cf. fact 9(9).

We can also derive the following corollary for the cases $n = 3, 4$:

COROLLARY 23 *Let $n = 3$ or 4. Then CRA_n is the class of all atomic algebras in RA_n. Hence, CRA_n is an elementary class.*

PROOF:

By proposition 17 and theorem 22 above, if $\mathcal{A} \in \mathsf{CRA}_n$ then $\mathcal{A} \in \mathsf{RA}_n$ and $\mathcal{A}$ is atomic. We check the converse. Fact 9(5), due to Maddux, shows that for $n = 3, 4$, if $\mathcal{A} \in \mathsf{RA}_n$ is atomic then there is an n-dimensional basis for $\mathcal{A}$ — the set of *all* atomic $\mathcal{A}$-networks with domain n forms one — so $\mathcal{A} \in \mathsf{B}_n$. By proposition 18, we conclude that $\mathcal{A} \in \mathsf{CRA}_n$.

CRA_n is an elementary class because RA_n is a variety (fact 9(7)), so elementary, and the property of being atomic is certainly elementary. $\square$

Note that boolean algebras behave similarly: a boolean algebra has a complete representation as a field of sets if and only if it is atomic [HH97b, theorem 5]. We will see later that for $n > 4$, CRA_n is not elementary.

3.2 Arbitrary n-square representations

We still let $3 \leq n \leq \omega$. We will prove:

THEOREM 24 *Let $\mathcal{A}$ be a non-associative algebra. Then $\mathcal{A} \in \mathsf{RA}_n$ if and only if $\mathcal{A}$ has an n-square relativised representation.*

First, the easy left-to-right direction.

LEMMA 25 *If $\mathcal{A} \in \mathsf{RA}_n$ then $\mathcal{A}$ has an n-square relativised representation.*

PROOF:

Let $\mathcal{A} \in \mathsf{RA}_n$. By definition 8, there is a (complete) atomic non-associative algebra $\mathcal{B} \supseteq \mathcal{A}$ with an n-dimensional basis. By proposition 18, $\mathcal{B} \in \mathsf{CRA}_n$, so that there is a (complete) n-square relativised representation M of $\mathcal{B}$. That is, $M \models T^n_{\mathcal{B}}$. But $T^n_{\mathcal{A}} \subseteq T^n_{\mathcal{B}}$, so the reduct of M to the language $\mathcal{A}$ is a model of $T^n_{\mathcal{A}}$ and an n-square relativised representation of $\mathcal{A}$. $\square$

We begin the proof of the converse with the following lemma.

LEMMA 26 *For any non-associative algebra $\mathcal{A}$, if $\mathcal{A}$ has an n-square relativised representation then $\mathcal{A}^+ \in \mathsf{CRA}_n$.*

PROOF:

Recall that $\mathcal{A}^+$ is the canonical embedding algebra of $\mathcal{A}$. See [JT52] for information here.

By assumption, $T^n_{\mathcal{A}}$ has a model. Take an ω-saturated model M of $T^n_{\mathcal{A}}$. (Such an M always exists if $T^n_{\mathcal{A}}$ is consistent, even if the language $\mathcal{A}$ is uncountable: see [Hod93, corollary 10.2.2], for example. Hodges' book gives more information on saturation.) We show that M yields a complete n-square relativised representation of $\mathcal{A}^+$, so that $\mathcal{A}^+ \in \mathsf{CRA}_n$.

The atoms of $\mathcal{A}^+$ can be identified with the ultrafilters of $\mathcal{A}$. Thus, we may define M as an $At\mathcal{A}^+$-structure M^+ with the same domain as M, by:

$$M^+ \models f(x,y) \iff M \models r(x,y) \text{ for all } r \in f,$$

for every ultrafilter f of $\mathcal{A}$. We may then expand M^+ to an $\mathcal{A}^+$-structure in the usual way: for $\rho \in \mathcal{A}^+$ and $x, y \in M^+$, we let

$$M^+ \models \rho(x,y) \iff M^+ \models f(x,y) \text{ for some atom (ultrafilter) } f \leq \rho \text{ in } \mathcal{A}^+.$$

Now $\mathcal{A} \subseteq \mathcal{A}^+$, up to isomorphism, and we see that if $r \in \mathcal{A}$ then the interpretations of r in M and M^+ are the same. For take $x, y \in M$ with $M \models 1(x, y)$, and let $f = \{s \in \mathcal{A} : M \models s(x, y)\}$. By lemma 12, f is an ultrafilter on $\mathcal{A}$. Then $M^+ \models f(x, y)$, and f is the unique ultrafilter of $\mathcal{A}$ with this property. Let $r \in \mathcal{A}$ be arbitrary. Then $\mathcal{A}^+ \models f \leq r$ is equivalent to $r \in f$. Now if $M \models r(x, y)$ then $r \in f$, so by definition, $M^+ \models r(x, y)$. Conversely, if $M^+ \models r(x, y)$ then by uniqueness of f we have $r \in f$, so $M \models r(x, y)$.

We check that $M^+ \models T^n_{\mathcal{A}^+}$. Most checks are routine, as in lemma 21; it helps to remember that by the above, the binary relation symbols $1, 1'$ appearing in some of the axioms of $T^n_{\mathcal{A}^+}$ mean the same in M^+ and M. We confine ourselves here to those validations that require saturation. These are the non-universal axioms in $T^n_{\mathcal{A}^+}$.

Consider first the axiom $\exists xy\, \rho(x, y)$ of $T^n_{\mathcal{A}^+}$, for non-zero $\rho \in \mathcal{A}^+$. Take an atom $f \leq \rho$. Let p be the type

$$p(x, y) = \{r(x, y) : r \in f\},$$

without parameters. As f is a filter, any finite subset of p is implied, in any model of the boolean part of $T^n_{\mathcal{A}}$, by a single formula of the form $r(x, y)$ for some (necessarily non-zero) $r \in f$. As $M \models \exists xy\, r(x, y)$ for all non-zero $r \in \mathcal{A}$, we see that p is consistent (finitely satisfied in M). As M is ω-saturated, p is realised in M: there are $a, b \in M$ with $M \models r(a, b)$ for all $r \in f$. Thus, $M^+ \models f(a, b)$, and so $M^+ \models \rho(a, b)$.

Finally, consider the 'n-square' axioms of $T^n_{\mathcal{A}^+}$. Let $i, j < k < n$, and take $a_0, \ldots, a_{k-1} \in M^+$ and $\rho, \sigma \in \mathcal{A}^+$ with $M^+ \models [\rho\,;\sigma](a_i, a_j) \wedge \bigwedge_{l,m<k} 1(a_l, a_m)$. We find $a_k \in M^+$ with $M^+ \models \rho(a_i, a_k) \wedge \sigma(a_k, a_j) \wedge \bigwedge_{l,m\leq k} 1(a_l, a_m)$.

By definition of M^+, there is an ultrafilter f of $\mathcal{A}$ with $f \leq \rho\,;\sigma$ (in $\mathcal{A}^+$) and $M \models r(a_i, a_j)$ for every $r \in f$. By complete additivity of ';' in $\mathcal{A}^+$, we may choose ultrafilters g, h of $\mathcal{A}$ with $g \leq \rho$, $h \leq \sigma$, and $f \leq g\,;h$, in $\mathcal{A}^+$. Then $r\,;s \in f$ for every $r \in g$, $s \in h$.

Consider the type

$$q(x) = \{r(a_i, x), s(x, a_j), 1(x, a_l) : r \in g, s \in h, l < k\}$$

with a single free variable, x, and the finitely many parameters $a_0, \ldots, a_{k-1} \in M$. Because g and h are filters, any finite subset of q is implied in M by a single formula of the form

$$\varphi(x, a_0, \ldots, a_{k-1}) = r(a_i, x) \wedge s(x, a_j) \wedge \bigwedge_{l<k} 1(x, a_l)$$

for some $r \in g$, $s \in h$. Now $r\,;s \in f$, so that $M \models [r\,;s](a_i, a_j)$. Because $M \models T^n_{\mathcal{A}}$, we have $M \models \exists x \varphi(x, a_0, \ldots, a_{k-1})$. So q is consistent.

By ω-saturation, there is $a_k \in M$ realising q. Then as $1^{M^+} = 1^M$ is reflexive and symmetric, $M^+ \models g(a_i, a_k) \wedge h(a_k, a_j) \wedge \bigwedge_{l,m\leq k} 1(a_l, a_m)$. It follows that $M^+ \models \rho(a_i, a_k) \wedge \sigma(a_k, a_j) \wedge \bigwedge_{l,m\leq k} 1(a_l, a_m)$, as required.

By definition of M^+, if $a, b \in M^+$ and $M^+ \models 1(a, b)$ then there is an atom f of $\mathcal{A}^+$ with $M^+ \models f(a, b)$. So by proposition 16, M^+ is a complete n-square relativised representation of $\mathcal{A}^+$. $\square$

Now we can prove the converse to lemma 25. If $\mathcal{A}$ has an n-square relativised representation, then $\mathcal{A}^+ \in \mathsf{CRA}_n$. By proposition 17, $\mathcal{A}^+ \in \mathsf{B}_n$. Since $\mathcal{A} \subseteq \mathcal{A}^+$ (up to isomorphism), and $\mathcal{A}^+$ is complete and atomic, we have $\mathcal{A} \in \mathsf{RA}_n$ by definition of RA_n. This completes the proof of theorem 24.

COROLLARY 27 (MADDUX, 1983) *The variety* RA_n *is canonical. Indeed,* $\mathcal{A} \in \mathsf{RA}_n \Rightarrow \mathcal{A}^+ \in \mathsf{CRA}_n$.

PROOF:

Use lemmas 25 and 26, and theorem 22. □

4 Games and bases

Here, we introduce the games discussed in the introduction, and prove the equivalence of Maddux's original definition of n-dimensional relational basis to the game-theoretic one. Then we introduce the n-dimensional Lyndon conditions of the introduction. Finally, we show how to use games to provide an explicit equational axiomatisation of RA_n for finite n.

The ordinal n remains fixed, with $3 \leq n \leq \omega$.

4.1 Definitions

Let $\mathcal{A}$ be an atomic non-associative algebra, and $r \leq \omega$.

DEFINITION 28 We define the game $G^n_r(\mathcal{A})$, with r rounds and played on (atomic) $\mathcal{A}$-networks of size at most n. There are two players, $\forall$ and $\exists$, who the reader will recall from the introduction are male and female, respectively.

If $r = 0$, $\exists$ wins by default without play taking place. Assume that $r > 0$. In round 0, $\forall$ selects an atom a from $\mathcal{A}$, and $\exists$ responds with a network $N = N_0$ with at most n nodes and containing nodes x, y with $N(x, y) = a$.

In each subsequent round t ($1 \leq t < r$), assume that the most recently created network in the game is N_{t-1}, and that $|N_{t-1}| \leq n$. Round t proceeds in three steps:

1. First, if $|N_{t-1}| = n$ then $\forall$ deletes a node from N_{t-1}. That is, he chooses some node $w \in N_{t-1}$ and defines the network N to be the subnetwork of N_{t-1} with domain $N_{t-1} \setminus \{w\}$. (He does this even if $n = \omega$.)

   If $|N_{t-1}| < n$ then there is no need to delete a node and he defines $N = N_{t-1}$.

2. $\forall$ then chooses nodes $x, y \in N$ and atoms $a, b \in \mathcal{A}$ with $a\,;b \geq N(x, y)$.

3. Player $\exists$ must respond with a network $N_t \supseteq N$ with at most n nodes and containing a node z with $N_t(x, z) = a$ and $N_t(z, y) = b$.

This completes the round.

The rounds are numbered $0, 1, 2, \ldots, t, \ldots$ for $t < r$. If $\exists$ is always able to form the new network (N_t) in each round (t), then she wins the play of the game. Otherwise, $\forall$ wins.

DEFINITION 29 We also define the game $G^n_r(\mathcal{A}, N)$ for any $\mathcal{A}$-network N of size at most n; it is the same as $G^n_r(\mathcal{A})$, except that in round 0, $\forall$ simply presents $\exists$ with the network N. Thus, this game is $G^n_r(\mathcal{A})$ but starting with $N_0 = N$, rather than building N_0 in the first round.

It may be helpful to think of the nodes of the networks $N_0, N_1, \ldots$ as movable 'pebbles'. There are n pebbles available altogether. At any time, all ordered pairs of pebbles in play are labelled by atoms of $\mathcal{A}$, so as to create a network with the pebbles as nodes. In each round, $\forall$ can bring into play a spare pebble, if any remain. If not, he may move a pebble already in play; doing so erases the labels on all ordered pairs involving it. He then specifies the labels on two of the ordered pairs involving the new or relocated pebble, and $\exists$ has to complete the labelling to yield a network. This is all done r times, and if $\exists$ survives she has won.

DEFINITION 30 A *strategy* for a player is a set of rules telling that player what to do in every situation. It can be formalised as functions but it is not helpful to do so here. We say that a strategy is *winning* for its owner if its use always results in a win for the owner however the opponent plays.

Now we connect the games to n-dimensional bases. Athough this result does not seem to be very widely known, we have not been able to track down its origin. (It is certainly not ours.)

PROPOSITION 31 *Let $\mathcal{A}$ be an atomic non-associative algebra. $\mathcal{A}$ has an n-dimensional basis iff $\exists$ has a winning strategy in the game $G^n_\omega(\mathcal{A})$.*

PROOF:

[sketch] For left to right, let $\mathcal{M}$ be an n-dimensional basis for $\mathcal{A}$. We present the following strategy for $\exists$ in $G^n_\omega(\mathcal{A})$: 'always play a network from $\mathcal{M}$'. It is clearly winning, so long as it can always be implemented. Let us check this. In round 0, let $\forall$ play the atom a. There is $N_0 = (n, \nu) \in \mathcal{M}$ with $\nu(0,1) = a$. So $\exists$ may simply play N_0 as her move in round 0. In a subsequent round t, let $N_{t-1} \in \mathcal{M}$ be the current network. Note that $|N_{t-1}| = n$, so $\forall$ has to remove a node from N_{t-1}. This leaves a network N with domain $n \setminus \{k\}$ for some $k < n$. He then chooses atoms $a, b \in \mathcal{A}$ and nodes $i, j \in N$ with $a\,;b \geq N(i,j)$. By definition of $\mathcal{M}$, there is $N_t \in \mathcal{M}$ with $N_t(i,k) = a$, $N_t(k,j) = b$, and $N_t \supseteq N$. $\exists$ may respond to $\forall$'s move with N_t.

For right to left, if $\exists$ has a winning strategy in $G^n_\omega(\mathcal{A})$ then let $\mathcal{M}$ be the set of all atomic $\mathcal{A}$-networks N with domain n such that $\exists$ has a winning strategy in $G^n_\omega(\mathcal{A}, N)$ (see definition 29). It is easily seen that $\mathcal{M}$ is an n-dimensional relational basis for $\mathcal{A}$.[6] Property 2 of definition 6 is easily seen to hold; let us check property 1. If $a \in At\mathcal{A}$, let $\forall$ play a as his move in round 0 of $G^n_\omega(\mathcal{A})$. Suppose that $\exists$'s response, using her winning strategy, is the network N. There are nodes $p, q \in N$ with $N(p,q) = a$. Define a map $f : n \to N$ by $f(0) = p$ and $f(i) = q$ for $0 < i < n$. Then define a network N' with domain n, by $N'(i,j) = N(f(i), f(j))$, for $i, j < n$. Evidently, N' is an atomic $\mathcal{A}$-network, and as is easily seen, $\exists$ has a winning strategy in $G^n_\omega(\mathcal{A}, N')$. Thus, $N' \in \mathcal{M}$ and $N'(0,1) = a$, proving property 1. $\square$

This proposition is particularly helpful, not only because it allows the use of the simple and intuitive technique of games but because it gives a natural way of *approximating* the existence of an n-dimensional basis for an algebra $\mathcal{A}$, namely by saying that $\exists$ has a winning strategy in the games $G^n_r(\mathcal{A})$ for larger and larger $r < \omega$. We will use it in section 5 as well as here.

[6] Actually, it is the largest such basis. Any fixed winning strategy for $\exists$ also yields an n-dimensional basis, and conversely.

4.2 n-dimensional Lyndon conditions

In this section and the next, we confine our attention to finite n ($3 \leq n < \omega$), the case $n = \omega$ having been handled in [HH97d, HH97b]. We can write down first-order sentences σ_r^n, which we call *n-dimensional Lyndon conditions,* expressing that $\exists$ has a winning strategy in the game $G_r^n(\mathcal{A})$, for atomic $\mathcal{A}$ and any $r < \omega$. We will then show that the class of non-associative algebras satisfying all the n-dimensional Lyndon conditions is precisely the closure of CRA_n under elementary equivalence.

The technique we use to obtain the σ_r^n is shown in [HH97a, HH97d]; we will not go into the details here. Formally:

PROPOSITION 32 *For any $r < \omega$, there is a first-order sentence σ_r^n such that for any atomic non-associative algebra $\mathcal{A}$, $\exists$ has a winning strategy in the game $G_r^n(\mathcal{A})$ if and only if $\mathcal{A} \models \sigma_r^n$.*

It is not true in general that if $\mathcal{A}$ is atomic and $\mathcal{A} \models \sigma_r^n$ for all $r < \omega$ then $\exists$ has a winning strategy in $G_\omega^n(\mathcal{A})$. For by proposition 31, $\exists$ has a winning strategy in $G_\omega^n(\mathcal{A})$ if and only if $\mathcal{A} \in \mathsf{B}_n$; so the above statement follows from theorem 2, which shows that B_n is not elementary. However, there is a relation between these two properties, given by the following lemma (adapted from [HH97b]).

LEMMA 33 *Let $\mathcal{A}_r$ ($r < \omega$) be atomic non-associative algebras, and let $3 \leq n < \omega$. Suppose that $\exists$ has a winning strategy in $G_r^n(\mathcal{A}_r)$ for each $r < \omega$. Let D be any non-principal ultrafilter on ω. Then $\exists$ has a winning strategy in $G_\omega^n(\mathcal{B})$, where $\mathcal{B}$ is the ultraproduct $\mathcal{B} = \prod_{i<\omega} \mathcal{A}_i/D$. Hence, $\mathcal{B} \in \mathsf{B}_n$.*

PROOF:

Roughly, the idea is to keep a 'large' (i.e., belonging to D) set of indices r such that $\exists$'s strategy is still running in $G_r^n(\mathcal{A}_r)$ and such that an ultraproduct of these $\mathcal{A}_r$ is isomorphic to $\mathcal{B}$.

The well-known ultraproduct theorem of Łoś (see, e.g., [Hod93, theorem 9.5.1]) tells us that for any first-order formula $\varphi(\bar{x})$ of the signature of non-associative algebras, if $\bar{a}_i$ is an $|\bar{x}|$-tuple of elements of $\mathcal{A}_i$ for each $i < \omega$, and $\bar{b} = \prod_i \bar{a}_i/D$ the corresponding tuple in $\mathcal{B}$, then $\mathcal{B} \models \varphi(\bar{b})$ iff $\{i < \omega : \mathcal{A}_i \models \varphi(\bar{a}_i)\} \in D$. Using this, we see for example that $\mathcal{B}$ is an atomic non-associative algebra.

The game $G_\omega^n(\mathcal{B})$ proceeds by forming atomic $\mathcal{B}$-networks $N^0, N^1, \ldots$, say. $\exists$'s strategy will be to ensure that at every stage $t < \omega$, there are algebras $\mathcal{A}_i^t$ ($i < \omega$) taken from the original $\mathcal{A}_i$, atomic $\mathcal{A}_i^t$-networks N_i^t ($i < \omega$), and a non-principal ultrafilter D^t on ω, such that:

- $\mathcal{B} \cong \prod_{i<\omega} \mathcal{A}_i^t/D^t$
- For all $i < \omega$, $dom(N_i^t) = dom(N^t) \subseteq n$, and $N^t(x,y) = \prod_{i<\omega} N_i^t(x,y)/D^t \in \mathcal{B}$ for all nodes x, y of N^t. For short, we write this last as $N^t = \prod_{i<\omega} N_i^t/D^t$.
- for each $i < \omega$, $\exists$ has a winning strategy in $G_i^n(\mathcal{A}_i^t, N_i^t)$ (defined in definition 29).

These conditions are easy to arrange for $t = 0$, as follows. Let $\forall$ begin $G_\omega^n(\mathcal{B})$ with the atom $a \in \mathcal{B}$. We may choose atoms $a_i \in \mathcal{A}_i$ ($i < \omega$) with $a = \prod_{i<\omega} a_i/D$. (By Łoś' theorem, a_i must be an atom for a large set of i (in the ultrafilter); and as the remaining a_i are arbitrary and the $\mathcal{A}_i$ are all atomic, we can replace them by atoms if need be.) For each $i > 0$, $\exists$ begins a 'private' game $G^n_i(\mathcal{A}_i)$ by taking $\forall$'s role and playing a_i. She then responds with her winning strategy, which she has by the lemma assumptions, to give an $\mathcal{A}_i$-network N_i of size at most n, containing nodes x_i, y_i such that $N^i(x_i, y_i) = a_i$. Note that $i > 0$, so the strategy allows her to make at least one move.

We may assume without loss of generality that the set of nodes of each N_i is a subset of n. As n is finite, it is now easily seen that there is $S \in D$ such that $0 \notin S$, all the N_i $(i \in S)$ have the same set of nodes, and for some x, y in this set of nodes, we have $x_i = x$ and $y_i = y$ for all $i \in S$.

Let $D' = \{X \cap S : X \in D\}$, a non-principal ultrafilter on S. We have $\mathcal{B} \cong \prod_{i\in S} \mathcal{A}_i/D'$. Define a $\mathcal{B}$-network N^0 with set of nodes N_i for any $i \in S$, and with labels given by $N^0(z, w) = \prod_{i\in S} N_i(z, w)/D'$, for all $z, w \in N^0$. This is easily seen to be an atomic $\mathcal{B}$-network. We have $N^0(x, y) = \prod_{i\in S} a_i/D' = a$.

Now take a (the) order-preserving bijection $\iota : \omega \to S$ and use it to renumber everything, as follows. For $i < \omega$, define $\mathcal{A}^0_i = \mathcal{A}_{\iota(i)}$, and $N^0_i = N_{\iota(i)}$. For each $i < \omega$, since $\exists$ used a winning strategy in one round of $G^n_{\iota(i)}(\mathcal{A}^0_i)$ to create N^0_i, she has a winning strategy in the game $G^n_{\iota(i)-1}(\mathcal{A}^0_i, N^0_i)$. (Her strategy is: 'continue with the strategy in progress'.) Because $0 \notin S$, we have $\iota(i) > i$ for all $i < \omega$, so that $\exists$ certainly has a winning strategy in the game $G^n_i(\mathcal{A}^0_i, N^0_i)$.

Now define $D^0 = \{\iota^{-1}(X) : X \in D'\}$, an ultrafilter on ω. Here, $\iota^{-1}(X) = \{i < \omega : \iota(i) \in X\}$. Then $\mathcal{B} \cong \prod_{i<\omega} \mathcal{A}^0_i/D^0$ and $N^0 = \prod_{i<\omega} N^0_i/D^0$. We have arrived at the situation claimed, and $\exists$ may play N^0 in response to $\forall$'s move in $G^n_\omega(\mathcal{B})$.

Assume the situation above for $t - 1$; we explain how $\exists$ can preserve it to t. The argument is very similar to the above. Let $\forall$ begin the tth round of $G^n_\omega(\mathcal{B})$ now. If $|N^{t-1}| = n$ he will delete a node, z, say, from the current network N^{t-1}. Let N' be the resulting network. (If $\forall$ does not delete a node we let $N' = N^{t-1}$.) $\forall$ now picks $x, y \in N'$ and atoms $a, b \in \mathcal{B}$ with $a ; b \geq N^{t-1}(x, y)$.

Let $a_i, b_i \in \mathcal{A}^{t-1}_i$ $(i < \omega)$ be atoms such that $a = \prod_{i<\omega} a_i/D^{t-1}$ and $b = \prod_{i<\omega} b_i/D^{t-1}$. By Łoś' theorem, there is $S \in D^{t-1}$ such that $a_i ; b_i \geq N^{t-1}_i(x, y)$ for every $i \in S$. We may assume that $0 \notin S$.

For each $i \in S$, let $\exists$ play the role of $\forall$ in the private game $G^n_i(\mathcal{A}^{t-1}_i, N^{t-1}_i)$, in which we assume she has a winning strategy. Note that $i > 0$. First, she deletes z, if $\forall$ did so in the main game, to yield a network $N'_i \subseteq N^{t-1}_i$. If $\forall$ did not delete a node, we let $N'_i = N^{t-1}_i$. This is a legal $\forall$-move, as $|N^{t-1}_i| = |N^{t-1}|$. Then she plays the nodes $x, y \in N'_i$ and atoms $a_i, b_i \in \mathcal{A}^{t-1}_i$; this is a legal move for $\forall$ in the game because $i \in S$ so $a_i ; b_i \geq N'_i(x, y)$. She responds with her winning strategy, to yield an $\mathcal{A}^{t-1}_i$-network N^t_i, say, with $N^t_i \supseteq N'_i$, and $N^t_i(x, z_i) = a_i$ and $N^t_i(z_i, y) = b_i$ for some $z_i \in N^t_i$. We may assume that $z_i < n$, so that the nodes of each N^t_i form a subset of n.

Now there is $S' \in D^{t-1}$ contained in S, and $z \in n$, such that $z_i = z$ for all $i \in S'$. So all the N^t_i $(i \in S')$ have the same set of nodes — N, say. Let $D^t = \{X \cap S' : X \in D^{t-1}\}$, a non-principal ultrafilter on S'. Then $\mathcal{B} \cong \prod_{i\in S'} \mathcal{A}^{t-1}_i/D^t$. Let N^t be the $\mathcal{B}$-network with nodes N, given by $N^t = \prod_{i\in S'} N^t_i/D^t$. This is easily checked (by Łoś' theorem) to be a valid atomic $\mathcal{B}$-network, with $N^t \supseteq N'$, $N^t(x, z) = \prod_{i\in S'} a_i/D^t = a$ and $N^t(z, y) = \prod_{i\in S'} b_i/D^t = b$. $\exists$ may respond to $\forall$'s move in the main game with N^t. Now, by renumbering items as before, she may recover the inductive conditions.

We see that, assuming the inductive hypotheses, $\exists$ can survive each round of the main game $G^n_\omega(\mathcal{B})$, and keep the hypotheses into the next round. So she can survive $G^n_\omega(\mathcal{B})$ forever. $\square$

We now obtain the following; cf. [HH97b].

PROPOSITION 34 *Let $\mathcal{A}$ be an atomic non-associative algebra, and let $3 \leq n < \omega$. The following are equivalent:*

1. *$\mathcal{A} \models \sigma^n_r$ for all $r < \omega$*
2. *there is $\mathcal{B} \equiv \mathcal{A}$ with $\mathcal{B} \in \mathsf{CRA}_n$ (here, '$\equiv$' denotes elementary equivalence).*

PROOF:

For $2 \Rightarrow 1$, if $\mathcal{B}$ is as shown in (2) then by proposition 17, $\mathcal{B}$ has an n-dimensional basis, so by proposition 31, $\exists$ can win $G^n_\omega(\mathcal{B})$. So certainly, she can win $G^n_r(\mathcal{B})$ for all $r < \omega$, and proposition 32 yields $\mathcal{B} \models \sigma^n_r$ for all r. As $\mathcal{A} \equiv \mathcal{B}$, we have $\mathcal{A} \models \sigma^n_r$ for all r, too.

Now we prove $1 \Rightarrow 2$. If $\mathcal{A} \models \sigma^n_r$ for all $r < \omega$, then by proposition 32, $\exists$ has a winning strategy in the game $G^n_r(\mathcal{A})$ for all $r < \omega$. By lemma 33, any non-trivial ultrapower $\mathcal{B}$ of $\mathcal{A}$ over ω has an n-dimensional basis. By proposition 18, $\mathcal{B} \in \mathsf{CRA}_n$. By Łoś' theorem, $\mathcal{A} \equiv \mathcal{B}$, and we are done. $\square$

We will show below that for $n \geq 5$, this is the closest we can get to axiomatising CRA_n — CRA_n is not elementary.

4.3 Explicit equational axiomatisation of RA_n

Now we will show how to use games to derive an explicit equational axiomatisation of RA_n for finite $n \geq 3$. In fact we will not be too explicit: those interested may recover the full details from [HH97d, HH97a], where the case $n = \omega$ is treated.[7] We remark that by refining the argument, a universal first-order (not equational) axiomatisation using only $n^2 + 3$ variables can be obtained.

We can suppose that $n \geq 5$, as Maddux proved RA_3 and RA_4 to be SA and RA, respectively (fact 9(3,4)), and these varieties are *defined* by finite sets of equations. In particular, we may assume that all algebras mentioned are relation algebras. So fix such an n.

Let $\mathcal{A}$ be a relation algebra. Define a partial $\mathcal{A}$-network (not necessarily atomic) to be a pair $N = (N_1, N_2)$ where $N_2 : N_1 \times N_1 \to \mathcal{A}$ is a *partial* map satisfying

1. $\mathrm{dom}(N_2)$ is a reflexive and symmetric binary relation on N_1,
2. $N_2(x, x) \leq 1'$ for all $x \in N_1$,
3. If $(x, y), (y, z), (x, z) \in dom(N_2)$ then $(N_2(x, y) \,;\, N_2(y, z)) \cdot N_2(x, z) \neq 0$.

[7] A more general theorem providing explicit recursive universal axiomatisations of 'PC'_Δ' classes [Hod93] satisfying some restrictions is proved in [HH98a]. Classes such as $\mathsf{RA}_n, \mathbf{S}\mathfrak{Ra}\mathsf{CA}_n, \mathsf{RRA}$, etc., are covered.

Subnetworks We write $(N_1, N_2) \subseteq (N_1', N_2')$ if (i) $N_1 \subseteq N_1'$, and (ii) whenever $x, y \in N_1$ then $N_2(x,y)$ is defined if and only if $N_2'(x,y)$ is defined too, and in that case, $N_2'(x,y) \leq N_2(x,y)$.

Our other notational conventions for partial networks will be much as for the atomic ones seen earlier.

Games Now $\forall$ and $\exists$ play a game $R_r^n(\mathcal{A})$ of length r on partial $\mathcal{A}$-networks. Initially, $\forall$ chooses any non-zero $s \in \mathcal{A}$, and $\exists$ has to respond with a network N_0 with nodes x, y such that $N_0(x,y)$ is defined and is s. In round $t < r$, with N_{t-1} given, $\forall$ chooses a non-empty subset $C \subseteq N_{t-1}$ of size less than n, such that $N_{t-1}(x,y)$ is defined for all $x, y \in C$. He also chooses nodes $x, y \in C$, and elements $r, s \in \mathcal{A}$.

$\exists$ has two choices. She may reject, and play N_t the same as N_{t-1} except that $N_t(x,y) = N_{t-1}(x,y) \cdot -(r\,;s)$. Or she may accept, by playing a partial network $N_t \supseteq N_{t-1}$ containing a node z such that

- $N_t(x,z) = r$ and $N_t(z,y) = s$
- $N_t(x,y) = N_{t-1}(x,y) \cdot (r\,;s)$
- $N_t(z,w)$ and $N_t(w,z)$ are defined for all $w \in C$.

Axiomatising RA_n. To do this, we need some facts.

1. If $\mathcal{A}$ is a countable relation algebra, then $\mathcal{A}$ has an n-square relativised representation iff $\exists$ has a winning strategy in the game $R_r^n(\mathcal{A})$ for all $r < \omega$. This can be proved using arguments similar to those in [HH97d, proposition 23] and [HH97a, proposition 5.2].

2. For each $r < \omega$, there is an $\forall_1$ first-order sentence ρ_r^n, effectively obtainable from n, r, such that for any relation algebra $\mathcal{A}$, $\exists$ has a winning strategy in $R_r^n(\mathcal{A})$ if and only if $\mathcal{A} \models \rho_r^n$ (use arguments from [HH97d, theorems 24, 25] or from [HH97a, proposition 5.4]).

Recall now (e.g., from [JT52]) that a relation algebra is simple if and only if it satisfies $\forall x(x \neq 0 \leftrightarrow 1\,;x\,;1 = 1)$.

3. For each r there is an equation ε_r^n, equivalent in simple relation algebras to ρ_r^n and effectively obtainable from ρ_r^n. This is well-known; see, e.g., [HH97d, lemma 27] or [HH97a, lemma 5.6].

4. Any relation algebra $\mathcal{A}$ is isomorphic to a subalgebra of a product relation algebra of the form $\prod_{s\in S} \mathcal{A}_s$, where each $\mathcal{A}_s$ is simple and a homomorphic image of $\mathcal{A}$. See [JT52, theorem 4.15].

So the following are equivalent for any *countable* relation algebra $\mathcal{A}$:

I. $\mathcal{A} \in \mathsf{RA}_n$

II. Each $\mathcal{A}_s \in \mathsf{RA}_n$ (by (4) above, since RA_n is a variety)

III. Each $\mathcal{A}_s$ has an n-square relativised representation (by theorem 24)

IV. $\exists$ has a winning strategy in the game $R^n_r(\mathcal{A}_s)$ for each s and all $r < \omega$ (by (1) above; note that the $\mathcal{A}_s$ must be, and are, countable)

V. $\mathcal{A}_s \models \rho^n_r$ for all s and all $r < \omega$ (by (2))

VI. $\mathcal{A}_s \models \varepsilon^n_r$ for all s and all $r < \omega$ (by (3))

VII. $\mathcal{A} \models \varepsilon^n_r$ for all $r < \omega$ (by (4)).

Now take an arbitrary relation algebra $\mathcal{A}$, and a countable elementary subalgebra $\mathcal{A}'$ of $\mathcal{A}$. Since RA_n is known to be elementary (fact 9(7)), $\mathcal{A}$ is in RA_n if and only if $\mathcal{A}'$ is. Hence, $\mathcal{A} \in \mathsf{RA}_n$ iff $\mathcal{A}' \in \mathsf{RA}_n$ iff $\mathcal{A}' \models \varepsilon^n_r$ for all $r < \omega$ (by the above), iff $\mathcal{A} \models \varepsilon^n_r$ for all $r < \omega$ (as $\mathcal{A}' \preceq \mathcal{A}$). We obtain:

THEOREM 35 *For $5 \leq n < \omega$, the equations ε^n_r $(r < \omega)$, together with the equations defining* RA, *axiomatise the variety* RA_n. *Hence,* RA_n *has a recursive equational axiomatisation.*

5 A 'rainbow' construction of non-associative algebras

We now move towards proving theorems 1 and 2. In this section, we will construct a certain non-associative algebra which will be easy to use for that end.

We will first define a slight variation of a well-known Ehrenfeucht–Fraïssé-style game. We will call it $EF^p_r(A, B)$, where $p, r \leq \omega$ are ordinals. It is played by $\forall$ and $\exists$ between two model-theoretic structures A, B in a relational signature. Then we will construct a non-associative algebra $\mathcal{A}_{A,B}$ from arbitrary A, B in a binary (relational) signature, and show in theorem 39 that a winning strategy for $\exists$ in $EF^p_r(A, B)$ is equivalent to one in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. This is a result that may be of general use: for example, results in [HH97b] follow easily from it.

In our work here, $\exists$ has a winning strategy at least in the game $EF^2_2(A, B)$ (or equivalently in $G^4_3(\mathcal{A}_{A,B})$), and under these conditions $\mathcal{A}_{A,B}$ will in fact be associative and hence a relation algebra.

Theorem 39 will give us freedom to create relation algebras where $\exists$ can win G^n_r for certain n, r but not others, just by varying the structures A, B. It will then be an easy matter to prove theorems 1 and 2, by choosing A, B carefully.

5.1 The Ehrenfeucht–Fraïssé game

DEFINITION 36 Let L be a signature consisting only of relation symbols (no function or constant symbols). Let A, B be L-structures, and let $p, r \leq \omega$. The game $\mathbf{EF}^p_r(A, B)$ is defined as follows. It is a game played by $\forall$, $\exists$, with r rounds and p pairs of 'pebbles'. At each stage of the game, some of the pebble pairs may be in play, the remainder being left on the side. If a pair is in play, then one pebble of the pair is placed on an element of A, and the other on an element of B. At any stage, $\forall$ may remove any number of pebble pairs in play and place them at the side. This does not count as a move in the game. (It is convenient to allow him to do this, even though it is evidently not in general to his advantage.)

The game has r rounds, numbered $0, 1, \ldots, i, \ldots$ for $i < r$. Initially there are no pebbles in play. In each round:

- $\forall$ picks up one pebble from a pair not in play, and places it on an element of A. (If all pebbles are in play at the beginning of the round, $\forall$ will first have to exercise his option to remove some.)

- $\exists$ then places the other pebble of that pair on an element of B.

This completes the round. $\forall$ wins at that point if the relation $R \subseteq A \times B$ defined by the pebble pairs is not a partial homomorphism from A to B. Here, 'homomorphism' is as usual in model theory and universal algebra:

DEFINITION 37 *A* partial homomorphism *from A to B is a partial map $f : A \to B$ such that for every k-ary relation symbol $P \in L$ and $a_1, \ldots, a_k \in dom(f)$, if $A \models P(a_1, \ldots, a_k)$ then $B \models P(f(a_1), \ldots, f(a_k))$.*

So formally, let R consist of all pairs (a, b) such that $a \in A$ and $b \in B$ are elements covered by pebbles in the same pair. $\exists$ loses unless R defines a partial homomorphism from A to B. If $\exists$ does not lose in this way, the game goes on for another round. If $\exists$ never loses in any round, then she wins the play.

EXAMPLE 38 As a simple (but useful) example, let A, B be finite complete undirected graphs without loops (reflexive edges). If $|A| > |B|$, then $\forall$ has a winning strategy in $\mathbf{EF}^p_r(A, B)$ for any $p, r > |B|$. In each round $0, 1, \ldots, |B|$, he places a new pebble on a new element of A. To avoid losing, $\exists$ must respond by placing the other pebble of the pair on an unused element of B, because B has no loops. After $|B|$ rounds there will be no such element, and she will lose at the next round. On the other hand, if $p \leq |B|$ then $\exists$ has a winning strategy for $\mathbf{EF}^p_\omega(A, B)$, because with only at most $|B|$ pebble pairs, there always will be an unused element of B.

We remark that there are logical characterisations of the game $\mathbf{EF}^p_r(A, B)$: if L and r are finite, $\exists$ has a winning strategy if and only if $A \models \sigma \Rightarrow B \models \sigma$ for every existential L-sentence σ written with at most p variables, possibly re-used, and of quantifier depth at most r. In the example above, a relevant sentence is $\exists x_0, \ldots, x_p \bigwedge_{i<j\leq p} E(x_i, x_j)$, where E is the edge relation (playing the role of $\neq$). A similar result for arbitrary L and $r = \omega$ can also be proved [Bar77].

5.2 The modified game $EF^p_r(A, B)$

The definition of the Ehrenfeucht–Fraïssé game $\mathbf{EF}^p_r(A, B)$, above, is quite standard. We need a minor modification of the game to make it appropriate to our purposes.

In the constructions that follow we will use a relational language L with only unary and binary predicates, and A, B will be structures for this language. We define a game over A, B where $\forall$ has to ensure that there are always at least two pebbles on distinct elements of A. So we require that $\forall$ *makes his first two moves together* and they must be distinct; and that if at any later time the number of distinct elements of A covered by pebbles falls below 2, then $\exists$ wins immediately and the game is terminated at that point. So if any of p, r, $|A|$ are less than two, we get a degenerate game in which $\forall$ cannot make a legal move and so he is certain to lose. If $p = 2$, $\forall$ can only hope to win after his combined first two rounds, for at that point (if $r > 2$) he must pick up a pebble.

Thus, in the combined round 0 and round 1, $\forall$ is required to place two pebbles on two distinct points $a, a' \in A$. $\exists$ has to place the two corresponding pebbles on points $b, b' \in B$ such that $\{(a, b), (a', b')\}$ is a partial homomorphism from A to B. If she cannot manage to do this then she loses immediately.

The play then continues as before, except that if at any stage $\forall$ exercises his option of removing some pairs of pebbles, and as a result of this either there are no pebbles left in play or all his remaining pebbles occupy a single point $a \in A$, then $\forall$ loses the play and the game is terminated.

Clearly, if $\forall$ has a winning strategy in $\mathbf{EF}_r^p(A,B)$ then he has a winning strategy in which he never places two pebbles on the same point $a \in A$. So we assume that the game rules additionally state that he always plays this way, whether or not he has a winning strategy. If he ever places a pebble on top of another (in A), he loses.

All other rules remain unchanged.

Let us call this modified game $EF_r^p(A,B)$ (the change in the notation, in case you missed it, is just that we do not use bold font for EF).

5.3 The algebra $\mathcal{A}_{A,B}$

Until the end of section 5, we fix structures A, B in a signature L consisting of only unary and binary relation symbols. We will define a non-associative algebra $\mathcal{A}_{A,B}$, by listing its atoms, stating their converses, and then defining composition on the atoms. We can then define $\mathcal{A}_{A,B}$ by taking the power set of this set of atoms, and define converse and composition by an infinitary distribution rule: that is, $\mathcal{A}_{A,B}$ will be the complex algebra over the atom structure we define. (Any subalgebra with the same atoms will also do.) The reader should not confuse the structure A with the domain of the algebra $\mathcal{A}_{A,B}$.

We will regard the atoms as coloured. The identity, $1'$, is an atom with no colour. Non-identity atoms are of five different colours: green, white, yellow, black, and red, and here they are:

$$
\begin{aligned}
\text{Green} &= \{\mathsf{g}_a : a \in A\} \\
\text{White} &= \{\mathsf{w}\} \cup \{\mathsf{w}_S : S \subseteq A, |S| \leq 2\} \\
\text{Yellow} &= \{\mathsf{y}\} \\
\text{Black} &= \{\mathsf{b}\} \\
\text{Red} &= \{\mathsf{r}_{bb'} : b, b' \in B\}
\end{aligned}
$$

The atoms shown are all distinct. Note that if A and B are finite, there are finitely many atoms and $\mathcal{A}_{A,B}$ will be finite.

All atoms are self-converse ($\breve{x} = x$) except the red atoms where we have $(\mathsf{r}_{bb'})^{\smile} = \mathsf{r}_{b'b}$.

Composition in an atomic non-associative algebra may be defined by listing all triples of atoms (x,y,z) such that $x\,;y \geq \breve{z}$ (compare [Lyn50, §4] and [Mad82, §2]). We shall call such triples *consistent triples,* or, more loosely, *consistent triangles.* If (x,y,z) is a consistent triple, then its Peircean transforms (y,z,x), (z,x,y), $(\breve{x},\breve{z},\breve{y})$, $(\breve{y},\breve{x},\breve{z})$, and $(\breve{z},\breve{y},\breve{x})$ will also be consistent. We define the consistent triples of atoms of $\mathcal{A}_{A,B}$ to consist of all triples *except for* Peircean transforms of the following, where the indices a, a' range over A and the indices $b, b', b_1, b_1', b_2, b_2', b_3, b_3'$ range over B.

$$
\begin{array}{rll}
(1', s, t) & \text{unless } s = \breve{t} \text{ (any atoms } s,t) & (1) \\
(g, g', g''), (g, g', W) & \text{any green atoms } g, g', g'' \text{ and white atom } W & (2) \\
(\mathsf{y},\mathsf{y},\mathsf{y}), (\mathsf{y},\mathsf{y},\mathsf{b}) & & (3) \\
(\mathsf{g}_a, \mathsf{y}, \mathsf{w}_S) & \text{unless } a \in S & (4) \\
(\mathsf{g}_a, \mathsf{g}_{a'}, \mathsf{r}_{b'b}) & \text{unless } \{(a,b),(a',b')\} \text{ is a partial homomorphism} & (5) \\
(\mathsf{r}_{b_1b_1'}, \mathsf{r}_{b_2,b_2'}, \mathsf{r}_{b_3b_3'}) & \text{unless } b_1' = b_2, b_2' = b_3 \text{ and } b_3' = b_1 & (6)
\end{array}
$$

We will refer to these rules as 'rules 1 to 6'. This completes the definition of $\mathcal{A}_{A,B}$. By [Mad82, theorem 2.2], it is a non-associative algebra.

Our aim is to prove:

THEOREM 39 *Let A, B be structures in a relational signature L consisting of only unary and binary relation symbols. Let $p, r \leq \omega$. Then $\exists$ has a winning strategy in the game $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ if and only if she has a winning strategy in the game $EF^p_r(A, B)$.*

Here, '+' denotes ordinal sum, so that $2 + \omega = \omega$, for example.

REMARK 40 It is not hard to show that for any atomic non-associative algebra $\mathcal{A}$, $\exists$ has a winning strategy in $G^4_3(\mathcal{A})$ if and only if $\mathcal{A}$ is associative (and hence a relation algebra). Cf. [Mad82, theorem 2.2.5]. In the current paper, $\exists$ has a winning strategy in $EF^2_2(A, B)$, so the algebra $\mathcal{A}_{A,B}$ will in fact be a relation algebra.

Theorem 39 will follow from corollary 44 and proposition 46 below, which prove the directions '$\Rightarrow$' and '$\Leftarrow$', respectively. The cases where $p < 2$ or $r < 2$ are degenerate and we make no use of them later.

First a definition and an observation.

DEFINITION 41 A *red clique* R is a possibly empty $\mathcal{A}_{A,B}$-network such that for any distinct elements m, n of R, the label $R(m, n)$ is a red atom.

The observation is that given any red clique R with at least two nodes, each node $n \in R$ can be given a unique index $\beta(n, R) \in B$ such that if $m, n \in R$ are distinct then $N(m, n) = \mathsf{r}_{\beta(m,R),\beta(n,R)}$. (Explicitly, we may let $\beta(n, R) = b$, where $N(n, m) = \mathsf{r}_{bb'}$ for some $m \in R$ and $b' \in B$; rule 6 shows that this is well-defined.) When the context is clear we will refer to the index of the node n simply as $\beta(n)$.

5.4 How $\forall$ can win $G(\mathcal{A}_{A,B})$

Before we show how $\forall$ can win this game we introduce a harmless assumption which applies for the rest of section 5. We will use this assumption later.

ASSUMPTION 42 In a play of some $G^n_r(\mathcal{A}_{A,B})$, suppose that in some round, t, $\forall$ picks an edge (m, n) from the current network N (N is either N_{t-1} or what remains of it after $\forall$ has removed a node), and atoms $s, t \in \mathcal{A}_{A,B}$. If there is already a node $l \in N$ such that $N(m, l) = s$ and $N(l, n) = t$, then $\exists$ does not need to extend the network N for her move — she may set $N_t = N$. There is no advantage to $\forall$ in making such a move, and we assume that he never does so.

PROPOSITION 43 *Suppose that $p, r \leq \omega$ and that $\forall$ has a winning strategy in $EF^p_r(A, B)$. Then he has a winning strategy in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$.*

PROOF:

We have seen that if either p or r is less than 2 then $EF^p_r(A, B)$ is degenerate and $\forall$ cannot win. So assume $p, r \geq 2$. Let $\forall$ start a play of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ with the following sequence of moves. In round 0, he picks the atom w. $\exists$ will respond with a network N_0, consisting (without loss of generality) of just two nodes, c, c', with $c \neq c'$ and $N_0(c, c') = \mathsf{w}$. In subsequent rounds, $\forall$ never moves these two nodes.

Next, $\forall$ starts a private play of $EF^p_r(A, B)$ to help him choose his subsequent moves in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. Suppose his winning strategy for $EF^p_r(A, B)$ tells him to place two pebbles on the (distinct) elements $a, a' \in A$ in the combined round 0 and round 1. Then in round 1 of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ he chooses the edge (c, c') and the atoms g_a, y, and in round 2 he chooses the same edge (c, c') and the atoms $\mathsf{g}_{a'}, \mathsf{y}$. Since $a \neq a'$, this forces $\exists$ to construct a strict network N_2 with distinct nodes c, c', n_0, n_1 labelled as in figure 1. She has to choose a label for the edge (n_0, n_1). This label cannot be the identity nor green or white (by rule 2 applied to the triangle (n_0, n_1, c)) nor yellow or black (by rule 3 on triangle (n_0, n_1, c')) so it must be red, say $\mathsf{r}_{bb'}$ for some $b, b' \in B$. (This shows why we had to modify the Ehrenfeucht–Fraïssé game. The nodes in a red clique do not have indices until there are at least two of them. In the play of $EF^p_r(A, B)$, $\exists$ does not have to decide where to place her pebbles in B until there are two distinct pebbles in A.) In $\forall$'s

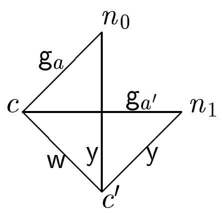


Figure 1: $\forall$'s first three moves

private play of $EF^p_r(A, B)$ he lets $\exists$ place two pebbles on b and b', corresponding to his pebbles on a, a'.

In round t $(2 < t < 1 + r)$ suppose, inductively, that the current network N_{t-1} has nodes $\{c, c', n_0, \ldots n_{q-1}\}$ for some $q < \omega$ with $2 \leq q \leq p$, and that for each $j < q$, $N_{t-1}(c', n_j) = \mathsf{y}$ and $N_{t-1}(c, n_j) = \mathsf{g}_{a_j}$ for some $a_j \in A$, the a_j being pairwise distinct. Thus, N_{t-1} is a strict network. As above, the labels on the edges (n_j, n_k) must be red (for $j, k < q$, $j \neq k$), so the nodes $\{n_j : j < q\}$ form a red clique of size at least two. Hence each node n_j has an index $\beta(n_j) \in B$. As part of an inductive hypothesis, suppose also that in round $t - 1$ of $EF^p_r(A, B)$, $\exists$ has not lost yet and $\forall$ is still using his winning strategy, and the situation corresponds to the situation in round t of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. That is, there are pairs of pebbles on $(a_j, \beta(n_j)) : j < q$.

If the number of pairs of pebbles q is already p, then $\forall$'s strategy will first direct him to remove a pair of pebbles, say the pair on $(a_j, \beta(n_j))$ for some $j < q$. In this case there must be $2+p$ nodes in the current network N_{t-1}, and $\forall$ removes the corresponding node n_j in the play of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. Assuming first that $p > 2$, removing the pair $(a_j, \beta(n_j))$ still leaves at least two distinct pairs of pebbles, so the play of $EF^p_r(A, B)$ continues. Similarly, removing n_j still leaves at least two nodes in the red clique, so they retain their indices.

The case $p = 2$ is slightly anomalous. We are assuming that $\forall$ has a winning strategy in $EF^p_r(A, B)$; but by removing a pair of pebbles he leaves only one pair in play and according to the rules he loses at this point. Hence we must assume that he has won $EF^p_r(A, B)$ already after rounds 0 and 1, and he never has to remove a pair of pebbles in the Ehrenfeucht–Fraïssé game when $p = 2$. We'll see below that this ensures that he has also won $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ at this point and that he never has to remove a node from a network in this game.

Returning to the general case ($p > 2$), let his winning strategy for $EF^p_r(A,B)$ then tell him to place a pebble on $a \in A$. In the play of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ he picks the edge (c,c') — as always — and the atoms g_a, y. That completes his move.

By the rules of $EF^p_r(A,B)$, a is distinct from all other points in A covered by pebbles, so this forces $\exists$ to add a new node n to the network, and as we have seen previously, n must be part of a red clique of size at least two. So n has an index $\beta(n) \in B$. In $\forall$'s private play of $EF^p_r(A,B)$, he lets $\exists$ place her corresponding pebble on $\beta(n)$.

In this way our inductive assumptions are maintained for another round.

Because $\forall$ has a winning strategy in $EF^p_r(A,B)$, there will come a time when $\forall$ places a pebble on $a \in A$ but there is nowhere in B for $\exists$ to place the other pebble while maintaining a partial homomorphism. At this point, no matter how $\exists$ has played in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$, the map $\{(a_j, \beta(n_j)) : j < q\}$ is not a partial homomorphism. Because the signature of A and B contains only unary and binary relations, this means that $\{(a_j, \beta(n_j)), (a_{j'}, \beta(n_{j'}))\}$ is not a partial homomorphism, for some $j < j' < q$. But this means that the triangle $(n_j, n_{j'}, c)$ is not consistent with rule 5, and so $\exists$ has lost in the game $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. $\square$

COROLLARY 44 *Suppose that $p, r \leq \omega$ and that $\exists$ has a winning strategy in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. Then she has a winning strategy in $EF^p_r(A,B)$.*

PROOF:

Assume that $\exists$ has a winning strategy in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. Clearly, $\forall$ can not also have a winning strategy in this game. By proposition 43, $\forall$ has no winning strategy in $EF^p_r(A,B)$. Let $\exists$ play $EF^p_r(A,B)$ according to the strategy: 'do not allow $\forall$ to get into a position from which he has a winning strategy'. The initial (empty) position satisfies this condition; and if in the current position, $\forall$ has no winning strategy, then whatever move $\forall$ makes, $\exists$ must obviously have a response to create a position from which he has no winning strategy. So she can implement this strategy throughout the game. She never loses (a lost position would certainly be one at which $\forall$ has a winning strategy), so this must be a winning strategy for her. (We have simply proved that the game $EF^p_r(A,B)$ is determined; cf. [GS53].) $\square$

5.5 How $\exists$ can win $G(\mathcal{A}_{A,B})$

In the proof of proposition 43, $\forall$ forced $\exists$ to attempt the game $EF^p_r(A,B)$ (and he won). He did this by building a red clique from a 'base' (c,c').

DEFINITION 45 *Let (c,c') be any edge in a strict network N (and $c \neq c'$). Define the* red clique based on (c,c') in N, *denoted $R_N(c,c')$, to be the subnetwork of N with nodes*

$$\{n \in N : N(c,n) \text{ is green and } N(c',n) \text{ is yellow}\}.$$

$R_N(c,c')$ may be empty, but it is certainly a red clique.

In proposition 46 below, we are going to prove the converse of proposition 43 — that if $\exists$ has a winning strategy in $EF^p_r(A,B)$ then she has one in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$. The idea behind the proof is not hard: if $\forall$ plays the same strategy used in proposition 43 then $\exists$ can win by using a winning strategy in $EF^p_r(A,B)$. The proof is complicated somewhat, because we cannot assume that $\forall$ plays in this way. We have to provide a winning $\exists$-strategy to cover every eventuality. But it turns out that $\forall$'s best strategy is more or less as described in proposition 43, in that the only way he could win would be to defeat $\exists$ on red cliques. So we will focus on red cliques in the proof here.

$\exists$ will associate an Ehrenfeucht–Fraïssé game with each red clique $R_N(c,c')$ of size at least two, where N is the current network, $c,c' \in N$ are distinct, and $N(c,c') \neq \mathsf{w}_S$ (for any S). These games will help her win the main game $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$.

The case where $N(c,c') = \mathsf{w}_S$ (some S) represents a degenerate clique. In a sense, such a clique is 'frozen', in that $\forall$ can only force a red clique of size $|S| \leq 2$ based on the edge (c,c'). From this position he can make no further progress.

PROPOSITION 46 *Suppose that $p,r \leq \omega$ and that $\exists$ has a winning strategy in $EF^p_r(A,B)$. Then she also has a winning strategy in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$.*

PROOF:

First let us deal with the case where either p or r is less than 2. By considering the rules of the game $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ we see that we will never get a network with more than three nodes. There is only one case where $\exists$ has to choose the label on an edge of such a network, and that is when we have a two-node network N with nodes n_0, n_1, say, and $\forall$ picks a reflexive edge, say (n_0,n_0), and a pair of atoms $a, \breve{a}$. $\exists$ has to add a new node n_2 and choose the label on the edge (n_1,n_2) (this determines the label on the edge (n_2,n_1)). But she can always do this, because whatever the values of $a, N(n_0,n_1)$ are, we have $s = \breve{a} ; N(n_0,n_1) \neq 0$; this may be checked using the definition of $\mathcal{A}_{A,B}$. $\exists$ may choose any atom $a \leq s$ to label (n_1,n_2). Thus, $\exists$ certainly has a winning strategy if either p or r is less than two, though the game is rather degenerate.

So assume that $p,r \geq 2$, and assume the hypotheses of the proposition. In each round of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$, $\exists$ plays using the following strategy.

In round 0, let $\forall$ play the atom a. If $a = 1'$, $\exists$ responds with the network N_0 with a single node, 0, with $N_0(0,0) = 1'$. If not, she plays the network N_0 with two nodes, $0,1$, say, with $N_0(0,0) = N_0(1,1) = 1'$, $N_0(0,1) = a$, and $N_0(1,0) = \breve{a}$. It is easily checked that N_0 is a strict atomic $\mathcal{A}_{A,B}$-network.

At the start of the tth round $(1 \leq t < 1+r)$, let the current network be N_{t-1}. We are going to describe how $\exists$ responds to a move of $\forall$ in this round, and to do this, we will again assume inductively that the network N_{t-1} has certain properties. We will check that these properties are preserved by $\exists$'s actions.

Some terminology will be very useful. If $\forall$ selects $(e,e') \in N_{t-1}$ and atoms $\varepsilon, \varepsilon'$, and if $\exists$ makes a proper extension $N_t \supset N_{t-1}$ with a new node v with $N_t(e,v) = \varepsilon$ and $N_t(v,e') = \varepsilon'$, then we say that the edges (e,v), (v,e), (e',v), (v,e'), and (v,v) of N_t (or their labels) are *chosen by* $\forall$. All other edges involving the new node v are said to be *chosen by* $\exists$. Edges not involving the new node v will have been chosen by $\forall$ or $\exists$ in earlier rounds on the same principles. All edges of the initial network N_0 are chosen by $\forall$.

Inductive assumptions on N_{t-1}

I. In the play so far $\exists$ has never chosen green or yellow atoms as labels for edges, nor the white atom w. (The only white atoms she ever chooses are of the form w_S $(S \subseteq A, |S| \leq 2)$.)

II. N_{t-1} is strict — if $m, n \in N_{t-1}$ are distinct then $N_{t-1}(m,n) \neq 1'$.

III. For each pair of distinct elements $c, c' \in N_{t-1}$ with $|R_{N_{t-1}}(c,c')| \geq 2$ (see definition 45 for the notation) and $N_{t-1}(c,c') \neq \mathsf{w}_S$ (any S), there is an associated private play of $EF^p_r(A,B)$, denoted $EF^p_r[c,c']$, in which $\exists$ is using her winning strategy. Further, there is a 1–1 correspondence between the pairs of pebbles currently in play in $EF^p_r[c,c']$ and the nodes of $R_{N_{t-1}}(c,c')$, such that for each such pair, if the elements of A, B covered by the pebbles in the pair are $a \in A, b \in B$ then the corresponding node $n \in R_{N_{t-1}}(c,c')$ satisfies

- $N_{t-1}(c,n) = \mathsf{g}_a$,
- $\beta(n, R_{N_{t-1}}(c,c')) = b$.

IV. There are at least as many rounds remaining in the play of any private $EF^p_r[c,c']$ as there are rounds remaining in the main game $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$.

V. N_{t-1} is a network.

We will show that $\exists$ has a strategy that can maintain all these assumptions to the next round. Our main aim is to prove inductive assumption V — this will show that $\exists$ has a winning strategy and prove the proposition. We need assumptions III and IV in order that we can define $\exists$'s strategy at all.

These properties hold trivially in N_0. Assuming they hold in N_{t-1} we must make sure they hold in N_t.

$\forall$'s move Let $\forall$ begin his move now. Suppose that he removes some node x from N_{t-1} to create a network $N \subseteq N_{t-1}$. At this point $\exists$ adjusts the private games so that the inductive assumptions still hold in N. This is mostly trivial to do, as follows. For any $x' \in N_{t-1}$, if $|R_{N_{t-1}}(x,x')| \geq 2$ then there is an associated play $EF^p_r[x,x']$. There are no winners in this game nor any survivors: this play is abandoned. Similarly, plays $EF^p_r[x',x]$ are also abandoned.

If $x \in R_{N_{t-1}}(c,c')$ (for some $c, c' \in N_{t-1}$) where there is an associated play $EF^p_r[c,c']$, then she gets $\forall$ to remove the pair of pebbles associated with x in the play $EF^p_r[c,c']$. If $|R_N(c,c')| < 2$ (because removing x from the red clique based on (c,c') brought the size of the clique below 2) then removing the associated pair of pebbles in $EF^p_r[c,c']$ leaves only one pair of pebbles in play. According to the rules, $\exists$ wins and this play is terminated. If $\forall$ does not remove a node from N_{t-1}, we define N to be N_{t-1}. In either case, we have arranged that our inductive assumptions still hold in N.

Then $\forall$ picks the edge (e,e') (say) from N, and the pair of atoms $\varepsilon, \varepsilon' \in \mathcal{A}_{A,B}$. So the triangle $(\varepsilon, \varepsilon', N(e',e))$ is consistent with the original rules of the definition of $\mathcal{A}_{A,B}$.

$\exists$'s response We are assuming that there is not already a node $z \in N$ such that $N(e,z) = \varepsilon$ and $N(z,e') = \varepsilon'$ (assumption 42). It follows that $\varepsilon, \varepsilon' \neq 1'$. Now

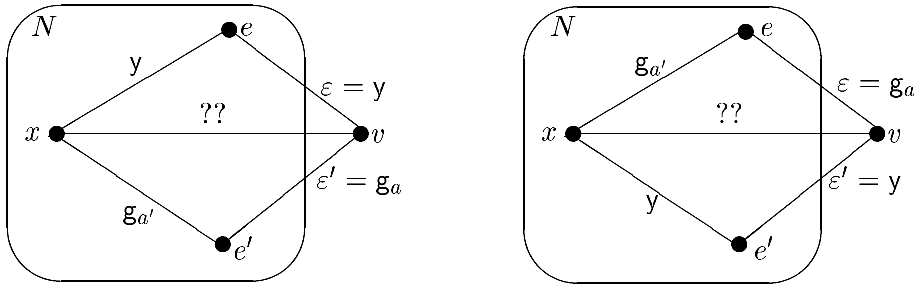


Figure 2: $\exists$ must adjoin v to the red clique containing x

$\exists$ forms N_t by adding to N a new node, v, defining $N_t(v,v) = 1'$, $N_t(e,v) = \varepsilon$, $N_t(v,e') = \varepsilon'$, and, necessarily, $N_t(v,e) = \breve{\varepsilon}$, and $N_t(e',v) = \breve{\varepsilon}'$. This is well-defined if $e = e'$, for then, the triangle $(\varepsilon, \varepsilon', 1')$ is consistent (i.e. it satisfies the rules defining composition for $\mathcal{A}_{A,B}$), so $\varepsilon = \breve{\varepsilon}'$ by rule 1 of the definition of $\mathcal{A}_{A,B}$.

Now $\exists$ has to choose labels for the new edges $(v,x),(x,v)$ (for $x \in N \setminus \{e,e'\}$), in such a way that the resulting graph forms an atomic network $N_t \supseteq N$ meeting the inductive assumptions above. The label $N_t(v,x)$ must always be the converse of $N_t(x,v)$, so she need only specify one of them. We will first state which labels $\exists$ chooses, then check that the inductive conditions are re-established. In particular we will check that they result in a consistent network N_t.

When choosing a label for (v,x), $\exists$'s first concern is that the triangles (v,x,e) and (v,x,e') are rendered consistent in N_t. The labels on (x,e) and (e,v) are already fixed, and she certainly cannot select a label for (v,x) that causes (v,x,e) to violate the rules, or N_t would not be a network. The same goes for (v,x,e').

First, she defines $N_t(v,x)$ for all those x satisfying conditions 1 or 2 below.

1. If there is no 'green path' from v to x via e or e' — that is, it is not true that ε and $N(e,x)$ are green or that ε' and $N(e',x)$ are green — then $\exists$ can choose a white atom for $N_t(v,x)$. She picks w_S, where

$$S = \{a \in A : \mathsf{g}_a \in \{\varepsilon, \varepsilon', N(e,x), N(e',x)\}\},$$

(By the case assumption, S has size at most two, so the definition makes sense.)

2. Otherwise, if there is no yellow path from v to x via e or e' — that is, it is not true that $\varepsilon = N(e,x) = \mathsf{y}$ or that $\varepsilon' = N(e',x) = \mathsf{y}$ — then $\exists$ defines $N_t(v,x) = \mathsf{b}$.

Now she considers all remaining $x \in N \setminus \{e,e'\}$, if any. For each such x, one of the situations shown in figure 2 applies — either the left or the right, and the same one for each x, because $\varepsilon, \varepsilon'$ are fixed for this round. The indices a, a' of the green atoms shown can be any elements of A. Assume that the second case shown (right) is the one that holds; the other situation is dealt with similarly (just swap e and e'). Thus, the set of x considered is

$$R_N(e,e') = \{x \in N : N(e,x) \text{ is green and } N(e',x) = \mathsf{y}\}.$$

We can suppose that $R_N(e,e') \neq \emptyset$, for otherwise there are no labels to choose here. There are three cases to consider:

3. $N(e,e') = \mathsf{w}_S$ for some $S \subseteq A, |S| \leq 2$.
   Take $x \in R_N(e,e')$ and let $N(x,e) = \mathsf{g}_{a'}$, as in the second diagram in figure 2. $\exists$ picks any two points $b,b' \in B$ such that $\{(a,b),(a',b')\}$ is a partial homomorphism from A to B. Such points b,b' certainly exist in B because we are assuming that $p,r \geq 2$ and that $\exists$ has a winning strategy in $EF^p_r(A,B)$. To label the edge (x,v) she chooses the atom $\mathsf{r}_{b'b}$.
4. $N(e,e') \neq \mathsf{w}_S$ (any S) and $|R_N(e,e')| = 1$.
   Then $R_N(e,e') = \{x\}$ for some x with $N(e,x) = \mathsf{g}_{a'}$ (for some $a' \in A$) and $N(e',x) = \mathsf{y}$. $\exists$ starts a new game $EF^p_r[e,e']$ by getting $\forall$ to place two pebbles on $a,a' \in A$. She uses her winning strategy in this game to place two corresponding pebbles on $b,b' \in B$. Then she labels the edge (x,v) with the atom $\mathsf{r}_{b'b}$.
5. $N(e,e') \neq \mathsf{w}_S$ and $|R_N(e,e')| > 1$.
   In this case, by our inductive assumptions, there is already an Ehrenfeucht–Fraïssé game $EF^p_r[e,e']$ in progress. Since $|N| < 2+p$ and $e,e' \notin R_N(e,e')$, we have $|R_N(e,e')| < p$, so fewer than p pairs of pebbles are in play in this game. $\exists$ picks up a spare pebble pair and, playing the role of $\forall$, places one of the pebbles in the pair on $a \in A$ (recall that $\varepsilon = \mathsf{g}_a$). She uses her winning strategy to respond by placing the other one on $b \in B$, say. Then for each $x \in R_N(e,e')$, she labels the edge (x,v) with the atom $\mathsf{r}_{\beta(x),b}$; here, $\beta(x)$ is of course $\beta(x, R_N(e,e'))$.

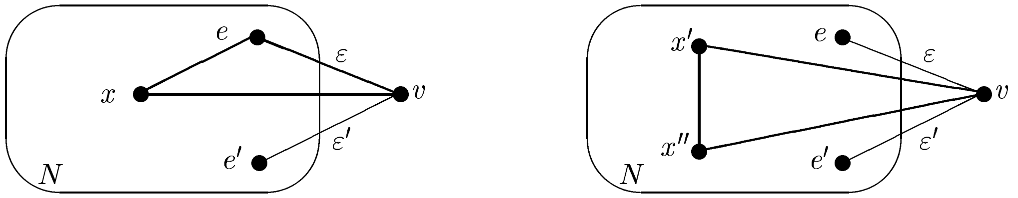
This completes the definition of N_t and of $\exists$'s strategy in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$

Inductive assumptions are still true Clearly $\exists$ never chose green or yellow atoms nor $\mathsf{w}, 1'$. This is enough to show that N_t is strict. So the inductive assumptions I and II hold for N_t.

Let us check inductive assumption III now. Let $R_{N_t}(c,c')$ be a red clique with $N_t(c,c') \neq \mathsf{w}_S$ (any S) and $|R_{N_t}(c,c')| \geq 2$. We have to check that there is an associated game $EF^p_r[c,c']$. If $R_{N_t}(c,c') \cup \{c,c'\} \subseteq N$, we already know that there is a suitable game in progress because the inductive assumptions hold in N. So we can suppose otherwise.

Now $c,c' \in N$. For assume not — say, $c \notin N$. Then c is the new node added in the current round (i.e., $c = v$), and at most two edges incident with c can have been chosen by $\forall$. Since $|R_{N_t}(c,c')| \geq 2$, there must be at least two nodes $n,m \in R_{N_t}(c,c')$ with $(c,n),(c,m)$ labelled with green atoms and $(c',n),(c',m)$ labelled by y. So $\exists$ must have chosen the labels on all other edges incident with c: in particular, she chose the label on (c,c'). The only green edges incident with c are (n,c) and (m,c); and since neither (n,c') nor (m,c') are green, there is no 'green path' from c to c'. Hence, according to $\exists$'s strategy (case 1), she would have chosen w_S for some S. But we are assuming that $N_t(c,c') \neq \mathsf{w}_S$. This is a contradiction, and we obtain $c \in N$. The proof that $c' \in N$ is similar.

So we can suppose that $R_{N_t}(c,c') \not\subseteq N$ and $c,c' \in N$. Thus, the new node v lies in $R_{N_t}(c,c')$. In the definition of $\exists$'s strategy, this situation is covered by cases 4 and 5: where $\forall$ chose the nodes $c,c' \in N$ with $N(c,c') \neq \mathsf{w}_S$, $\exists$ has to



(a) Primary triangle (x, e, v) (b) Secondary triangle (x', x'', v)

Figure 3: the triangles to check.

choose a red atom to label an edge (x, v), and where $|R_N(c, c')|$ is 1 or more than 1 respectively. In these two cases, her strategy was designed precisely to maintain inductive assumption III.

Inductive assumption IV is easy. Until round 2 is complete in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ (when there are four nodes in the network N), there can be no red clique $R_N(c, c')$ of size two or more. At the end of round 2, it may be that such a clique comes into being and that a game $EF^p_r[c, c']$ is started. This would involve playing the combined rounds 0 and 1 of this game. That still leaves $r - 2$ rounds left of $EF^p_r[c, c']$ (or ω rounds if $r = \omega$) which is the same as the number of rounds $1 + r - 3$ remaining in $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$ (or ω if $r = \omega$). So at this stage the inductive assumption is true. In each subsequent round of $G^{2+p}_{1+r}(\mathcal{A}_{A,B})$, at most one round of any existing game $EF^p_r[c, c']$ is played, and therefore this inductive assumption will remain true. Games resulting from cliques formed after round 2 have even more rounds remaining, so the inductive assumption holds for them, too.

N_t is a network For the resulting N_t to be a network, it is sufficient that every triangle (x, x', v), for distinct $x, x' \in N$, is consistent. There are three kinds of triangle of this form: the triangle (e, e', v) (if $e \neq e'$), which $\forall$ chose and so is guaranteed consistent; 'primary' triangles, which involve e or e' and one other node of N, so that $\exists$ had to define only one of the labels on the edges of the triangle; and finally 'secondary' ones involving neither of e, e', in which she had to define two labels. See figure 3.

$\exists$'s first concern has been that all primary triangles created were consistent, and it is mostly evident that she was successful. Rules 2 and 3 are not violated in primary triangles, since her strategy never selects a green or yellow atom to label an edge (v, x), and only chooses a white (respectively, black) label when there is no green (respectively, yellow) path from v to x. Her indices for white atoms were chosen precisely to conform with rule 4 in primary triangles.

$\exists$ only chooses a red atom if the two atoms chosen by $\forall$ were green and yellow. Thus rule 6 cannot be broken on primary triangles. Rule 5 does require a little checking. Suppose there is a primary triangle (v, w, x) (for some $w \in \{e, e'\}$, $x \in N \setminus \{e, e'\}$) whose three edges are labelled g_a, $\mathsf{r}_{b'b}$, and $\mathsf{g}_{a'}$, for some $a, a' \in A$, $b, b' \in B$. $\exists$ chose the label on (v, x). As she never chooses green atoms to label edges, she must have chosen the red label $\mathsf{r}_{b'b}$. So $\forall$ must have selected the edge $(w, w') \in \{(e, e'), (e', e)\}$ and a green and a yellow atom for his move. In this case, if $N(w, w') \neq \mathsf{w}_S$, $\exists$'s strategy ensures that $\{(a, b), (a', b')\}$ is part of a position in $EF^p_r[w, w']$ in which she is using her winning strategy (see cases 4 and 5 of $\exists$'s

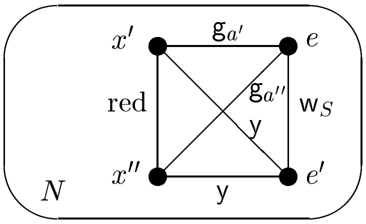


Figure 4: the subnetwork of N with nodes e, e', x', x''.

strategy), so that the map $\theta = \{(a, b), (a', b')\}$ is a partial homomorphism from A to B. If $N(w, w') = \mathsf{w}_S$ (some S) then case 3 of her strategy still ensures that θ is a partial homomorphism. Hence the triangle (v, w, x) is consistent.

Moreover, all secondary triangles are consistent. We show this as follows. Note first that any secondary triangle has two edges coloured by $\exists$, hence two edges labelled by white, black, or red atoms. Inspection of the original definition of consistent triples of atoms of $\mathcal{A}$ shows that in such a case, inconsistency is only possible if all three edges are red, rule 6 being the rule at risk.

So let the new node be v as usual, and let (x', x'', v) be a secondary triangle with each edge labelled by a red atom. $\exists$ chose two red atoms as labels for (v, x') and (v, x''). She would only do this if $\forall$ picked the edge (e, e') (say) and the atoms $\varepsilon = \mathsf{g}_a$ (for some $a \in A$) and $\varepsilon' = \mathsf{y}$ (the other case, where he picked (e', e), is symmetrical).

We claim that $N(e, e') \neq \mathsf{w}_S$ for any $S \subseteq A$. To see this, suppose for contradiction that $N(e, e') = \mathsf{w}_S$ for some $S \subseteq A$. Let the labels of the edges $(e, x'), (e, x'')$ be $\mathsf{g}_{a'}, \mathsf{g}_{a''}$, respectively. By rule 4 and the consistency of the triangles (e, e', x'), (e, e', x''), and (e, e', v'), we know that $\{a, a', a''\} \subseteq S$. Since $|S| \leq 2$, two of a, a', a'' must be equal to each other.

If a is equal to either a' or a'' then $\forall$ has broken assumption 42 by making a move which does not require a proper extension to the network. So suppose otherwise, so that $a' = a''$. We'll see that this also involves a violation of assumption 42. Consider the subnetwork of N with nodes x', x'', e, e' (see figure 4). Which of these four nodes was added most recently in the play of the game? Whichever it is, it should be incident with at most two edges within the subnetwork that were labelled by $\forall$; the rest must have been labelled by $\exists$. If it was x', then $\forall$ must have chosen the edge (x', e) labelled by $\mathsf{g}_{a'}$ and the yellow edge (x', e'), as $\exists$ never chooses these colours. But this breaks assumption 42, because in the round when x' was created, $\exists$ did not actually need to add the node x' as there was already the node x'' available. Similarly, x'' cannot have been the most recently added node (because x' was already there). If the most recent node was e, then in the round when e was added, $\forall$ labelled the edges (x', e) and (x'', e) with the atom $\mathsf{g}_{a'}$. Consequently, $\exists$ must have chosen the label on (e, e'). This is also true if e' was the last node to be built, by similar reasoning. In either case, her strategy would have directed her to choose the atom $\mathsf{w}_{\{a'\}}$ to label (e, e'), as there was no green path from e to e' and the two yellow-green paths both involved the atoms

y and $\mathsf{g}_{a'}$. Thus, $N(e,e') = \mathsf{w}_{\{a'\}}$. But then, by the consistency of $\forall$'s triangle (e,e',v), we must have $a \in S$; so $a = a'$, a contradiction to our assumption. This proves the claim.

So $N(e,e') \neq \mathsf{w}_S$ (any S). Since $\exists$ chose red atoms for (v,x') and (v,x''), it must be that $N(e,x') = \mathsf{g}_{a'}$, $N(e,x'') = \mathsf{g}_{a''}$ (some $a', a'' \in A$), and $N(e',x') = N(e',x'') = \mathsf{y}$. Thus, we have $x', x'', v \in R_{N_t}(e,e')$. Clearly, $\exists$ used part 5 of her strategy to assign a label to the edges (x',v) and (x'',v) of N_t — she will have defined $N_t(x',v) = \mathsf{r}_{\beta(x'),b}$ and $N_t(x'',v) = \mathsf{r}_{\beta(x''),b}$, for some $b \in B$ determined by her strategy in $EF_r^p[e,e']$. Here, $\beta(x')$ in full is $\beta(x', R_N(e,e'))$, and similarly for $\beta(x'')$. By definition of the indices $\beta(x'), \beta(x'')$, we have $N_t(x',x'') = N(x',x'') = \mathsf{r}_{\beta(x'),\beta(x'')}$. Thus, rule 6 is not violated.

We see that the triangle chosen by $\forall$, the primary triangles and the secondary triangles are all consistent with rules 1 to 6. This shows that N_t is a network. $\square$

Corollary 44 and proposition 46 yield theorem 39.

6 RA_{n+1} is not finitely based over RA_n

We can now harvest the fruits of our hard work, by easily proving theorems 1 and 2.

Let L be the signature consisting of a single binary relation symbol, '$<$'. For any irreflexive linear order I, and any $p \leq \omega$, let the L-structure $M[p,I]$ be the disjoint union of I and of a complete undirected graph K_p without loops, with p nodes. The relation symbol $<$ is interpreted in $M[p,I]$ as

$$<^{M[p,I]} = <^I \cup <^{K_p} \cup (I \times K_p) \cup (K_p \times I).$$

Warning: $<^{K_p}$ is the edge relation on K_p, and so is symmetric.

Now fix n, r with $4 \leq n, r < \omega$.

- Define $A[n,r] = M[n-3, 2^{r-1}]$, and $B[n,r] = M[n-3, 2^{r-1}-1]$. Here, 2^{r-1} and $2^{r-1}-1$ are regarded as ordered ordinals.
- As n and r are fixed, for brevity we write A for $A[n,r]$ and B for $B[n,r]$.
- Write $I(A)$ for the linear order part of A, $K(A)$ for the graph part, and similarly define $I(B), K(B)$.
- Define $\mathcal{A}_r^n$ to be the non-associative algebra $\mathcal{A}_{A,B}$, as defined in section 5.3.

See figure 5.

The next two lemmas are easier to prove if $\exists$ treats $\forall$'s initial combined two moves as two separate moves. This can only make it harder for her to win, so showing that she has a winning strategy in this situation is certainly good enough.

LEMMA 47 *$\exists$ has a winning strategy in the game $G_\omega^n(\mathcal{A}_r^n)$.*

PROOF:

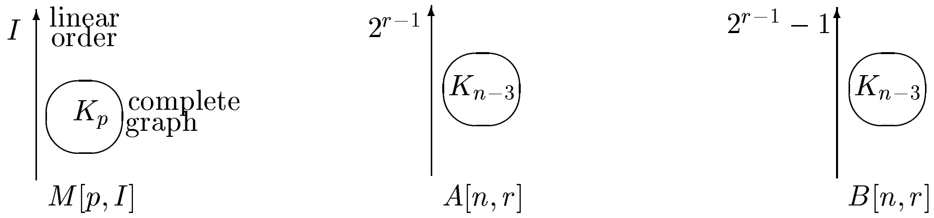


Figure 5: the structures $M[p, I]$, $A[n, r]$, $B[n, r]$.

By theorem 39 it suffices to show that $\exists$ has a winning strategy in the game $EF_\omega^{n-2}(A, B)$. $\exists$ plays in this game according to the following strategy. In defining it, we can suppose that $\forall$ has not fallen foul of the rule that at least two elements of A must always be pebbled, for otherwise, $\exists$ wins immediately and we are done.

Let $\forall$ pick up a spare pebble pair. Suppose that he places the first pebble of the pair on $a \in A$. The rules of $EF_\omega^{n-2}(A, B)$ demand that a is not currently occupied by a pebble. $\exists$ has to choose which element $a' \in B$ to put the other pebble on.

1. If possible, $\exists$ chooses a' to be an unoccupied element in $K(B)$.
2. If this is not possible, because all elements of $K(B)$ are occupied by pebbles, she chooses a' to be an arbitrary element $x \in I(B)$.

Because there are only $n-2$ pebble pairs, $\exists$ can always implement this strategy. Because the substructure S of B with domain $K(B) \cup \{x\}$ is isomorphic to K_{n-2}, so that $B \models y < y'$ for any distinct $y, y' \in S$, it follows that the pebble positions always define a homomorphism from A to B, and she wins. $\square$

LEMMA 48 *$\exists$ has a winning strategy in the game $G_r^{n+1}(\mathcal{A}_r^n)$.*

PROOF:

By theorem 39, it is enough if $\exists$ has a winning strategy in $EF_{r-1}^{n-1}(A, B)$.

The rounds of this game are numbered $i = 0, 1, \ldots, r-2$. Notation: at each round, write l for the number of rounds remaining: so $l = r - 1 - i$. If $X \subseteq I(B)$, an *interval* of $I(B)$ generated by X is a maximal convex subset of $I(B)$ disjoint from X.

$\exists$'s strategy is simply to keep the following conditions:

1. If one pebble of a pair is on an element of $K(A)$, the other is on an element of $K(B)$.
2. Similarly for $I(A)$, $I(B)$.
3. The partial map : $A \to B$ induced by the pebble positions is well-defined and one-to-one.
4. The partial map : $I(A) \to I(B)$ given by the positions of pebbles in the ordered part of A, B is order-preserving.
5. Every interval of $I(B)$ generated by her current pebbling contains at least $2^l - 1$ points.

These conditions are true initially, since there are no pebbles in play, $l = r - 1$, and the unique interval of $I(B)$ generated by the current pebbling, namely $I(B)$, contains $2^{r-1} - 1$ points. They clearly remain true if $\forall$ chooses to remove some pairs of pebbles.

Suppose they hold just before round i ($0 \leq i < r-1$). In round i, $\forall$ may place a pebble on an (unoccupied) element of $K(A)$. $\exists$ responds by placing the other pebble of the pair on an unoccupied element of $K(B)$. By the conditions, there will always be such an element. Or he may place a pebble in some interval of $I(A)$ generated by the other pebbles in play in $I(A)$. The corresponding interval of $I(B)$ (this notion is well-defined by condition 4) has at least $2^l - 1 = (2^{l-1} - 1) + 1 + (2^{l-1} - 1)$ points (note that $l \geq 1$ here), so $\exists$ may find a point in the interval, 'near the middle', such that the two new intervals it generates have at least $2^{l-1} - 1$ points. $\exists$ places the other pebble of the pair on such a point. This preserves the listed conditions into the next round. Clearly, this strategy wins $EF^{n-1}_{r-1}(A, B)$ for $\exists$. $\square$

LEMMA 49 *$\forall$ has a winning strategy in the game $G^{n+1}_{\omega}(\mathcal{A}^n_r)$.*

PROOF:

It suffices to show that $\forall$ has a winning strategy in $EF^{n-1}_{\omega}(A, B)$. Recall that A is the union of a complete graph $K(A)$ of size $n - 3$ and an ordinal $l = 2^{r-1}$, regarded as a linear order. $\forall$ uses the first $n - 1$ rounds to cover $K(A)$ and the elements $l-1, l-2 \in I(A)$ (i.e., the top two elements of $I(A)$) with $n-1$ pebbles. Because at least two out of every three distinct elements here are related by $<$ both ways, if $\exists$ is not to lose she must respond by pebbling $K(B) \cup \{b, b'\}$, for some $b, b' \in I(B)$. (All her choices must be distinct, and she cannot use three distinct elements of $I(B)$ as it is not true that two of these are related by $<$ both ways.)

Consider now the pebbles that $\exists$ placed on $b, b' \in I(B)$. Let the other pebbles of the pairs that these came from be placed on $a, a' \in A$, respectively. If $\exists$ has not yet lost, we have $a, a' \in I(A)$. For if not, $A \models a < a' < a$, but $B \not\models b < b' < b$, so that $\{(a, b), (a', b')\}$ is not a partial homomorphism from A to B. So at the end of round $n - 2$, assuming that he has not yet won, we see that $\forall$ has arranged that two elements of $I(A)$ are pebbled, the corresponding pebbles in B being in $I(B)$.

He will keep this condition from now on. Say that just after round i, $a_i < a'_i$ are pebbled in $I(A)$. If the corresponding elements of B are b_i, b'_i, then $b_i < b'_i$ in $I(B)$, or $\exists$ has lost.

$\forall$ can now force $\exists$ to play a two-pebble game of length ω on $I(A), I(B)$, which he can win because $I(A)$ is larger than $I(B)$. He simply preserves the condition that if he has not won yet, then

$$(*)_i \qquad b_i < a_i \text{ as ordinals,}$$

for each $i \geq n - 2$. This is true initially because $a_{n-2} = l - 2$ (by $\forall$'s choice) and $b_{n-2} = \min(b, b') \leq l - 3$ (because $|I(B)| = l - 1$). In each further round i, assuming that $(*)_{i-1}$ holds, he picks up the pebble on a'_{i-1} and moves it to $a_{i-1} - 1$. (As $n \geq 4$, this does not cause the number of pebble pairs in play to fall below 2.) So $a_i = a_{i-1} - 1$, $a'_i = a_{i-1}$. $\exists$ must pick up the other pebble of the pair, and move it to some element b_i of $I(B)$ below b_{i-1}, or she loses. By $(*)_{i-1}$,

$$b_i \leq b_{i-1} - 1 < a_{i-1} - 1 = a_i,$$

so $(*)_i$ holds.

Thus, eventually, b_i will be zero, a_i will not be, and $\forall$ will win after one more round. $\Box$

We can now prove theorem 1.
PROOF:

For any finite $n \geq 4$ and any finite $r \geq 4$, we have:

- $\mathcal{A}_r^n \in \mathsf{RA}_n$. For by lemma 47 and proposition 31, $\mathcal{A}_r^n \in \mathsf{B}_n \subseteq \mathsf{RA}_n$.
- $\mathcal{A}_r^n \notin \mathsf{RA}_{n+1}$. This is because by lemma 49 and proposition 31, $\mathcal{A}_r^n \notin \mathsf{B}_{n+1}$. As $\mathcal{A}_r^n$ is a finite algebra, by fact 9(9) we obtain $\mathcal{A}_r^n \notin \mathsf{RA}_{n+1}$.

Let $\mathcal{B}$ be the ultraproduct $\prod_{i<\omega} \mathcal{A}_{i+4}^n / D$, for any non-principal ultrafilter D on ω. By lemmas 48 and 33, $\exists$ has a winning strategy in $G_\omega^{n+1}(\mathcal{B})$, so $\mathcal{B} \in \mathsf{B}_{n+1} \subseteq \mathsf{RA}_{n+1}$.

It follows that RA_{n+1} is not finitely based over RA_n. Assume for contradiction that there is a first-order sentence σ such that for any non-associative algebra $\mathcal{A}$, $\mathcal{A} \in \mathsf{RA}_{n+1} \iff \mathcal{A} \in \mathsf{RA}_n$ and $\mathcal{A} \models \sigma$. Since $\mathcal{B} \in \mathsf{RA}_{n+1}$, so that $\mathcal{B} \models \sigma$, by Łoś' theorem [Hod93, theorem 9.5.1] we have $\mathcal{A}_r^n \models \sigma$ for infinitely many $r < \omega$. Take any such r. Then as $\mathcal{A}_r^n \in \mathsf{RA}_n$, we obtain $\mathcal{A}_r^n \in \mathsf{RA}_{n+1}$, a contradiction. $\Box$

We can also prove theorem 2.
PROOF:

A similar choice of algebras works; we only sketch the details here. Let $A = M[n-3, \mathbb{Z}]$ and $B = M[n-3, \mathbb{N}]$. One can show that

- $\exists$ has a winning strategy in $G_\omega^n(\mathcal{A}_{A,B})$ (cf. lemma 47; no change to the proof is required).
- $\exists$ has a winning strategy in $G_r^{n+1}(\mathcal{A}_{A,B})$ for all $r < \omega$ (cf. lemma 48; the inductive condition in the proof that all generated intervals of $I(B)$ have at least $2^l - 1$ points, where l is the number of rounds remaining, is easy to arrange, since $I(B)$ is infinite).
- $\forall$ has a winning strategy in $G_\omega^{n+1}(\mathcal{A}_{A,B})$. He plays as in lemma 49, forcing the two pebbles down the order $\mathbb{Z}$, move by move. $\exists$ cannot match this for ω rounds, because $\mathbb{N}$ is well-founded. So he will win.

Thus $\mathcal{A}_{A,B} \in \mathsf{B}_n \setminus \mathsf{B}_{n+1}$, but by lemma 33, any non-trivial ultrapower $\mathcal{B}$ of $\mathcal{A}_{A,B}$ over ω lies in B_{n+1}. Since $\mathcal{A}_{A,B} \equiv \mathcal{B}$, we see that B_{n+1} is not an elementary class for any finite $n \geq 4$, even 'with respect to the algebras in B_n only'. $\Box$

6.1 Summary of inclusions between classes

Finally, we ascertain the relations between some of the chief classes considered in the paper. See fact 9 for the analogues for RA_n.

THEOREM 50 $\mathsf{CRA} \subset \mathsf{CRA}_\omega = \mathsf{B}_\omega \subset \bigcap_{n<\omega} \mathsf{B}_n = \bigcap_{n<\omega} \mathsf{CRA}_n$, *where* $\subset$ *denotes strict inclusion.*

PROOF:

Clearly, any complete classical representation is a complete κ-square relativised representation for any cardinal κ. Hence $\mathsf{CRA} \subseteq \mathsf{CRA}_\omega$. To show that the inclusion is proper, define the non-associative algebra $\mathcal{A}$ to be $\mathcal{A}_{\mathbb{R},\mathbb{Q}}$, where $\mathbb{R}$ denotes the ordered real numbers and $\mathbb{Q}$ the ordered rationals. As $\mathbb{Q}$ is dense, it is clear that $\exists$ has a winning strategy in $EF^\omega_\omega(\mathbb{R}, \mathbb{Q})$. (The proof is similar to that of lemma 48, and we omit it.) By theorem 39, $\exists$ has a winning strategy in $G^\omega_\omega(\mathcal{A})$, and by proposition 31, $\mathcal{A} \in \mathsf{B}_\omega$. By theorem 22, $\mathcal{A} \in \mathsf{CRA}_\omega$.

But now, one may show using the method of proposition 43 that in any complete representation of $\mathcal{A}$ there will be an uncountable red clique (indexed by $\{\mathsf{g}_i : i \in \mathbb{R}\}$). Because $\mathbb{Q}$ is countable, it is easily seen that there is no uncountable red clique. We deduce that $\mathcal{A}$ has no complete representation, so $\mathcal{A} \notin \mathsf{CRA}$.

We proved that $\mathsf{CRA}_\omega = \mathsf{B}_\omega$ in theorem 22.

Evidently, $\mathsf{B}_\omega \subseteq \bigcap_{n<\omega} \mathsf{B}_n$, To see that the inclusion is proper, we use an example similar to one given in [HH97b]. Let $A = K_\omega$, a countably infinite complete undirected graph, and let B be the disjoint union of complete graphs of size n for all finite n: $B = \bigcup_{n<\omega} K_n$. We regard A and B as structures in the signature consisting of a single binary relation symbol, interpreted as the edge relation. Then evidently, $\exists$ has a winning strategy in $EF^n_\omega(A, B)$ for all finite n, as she may always respond to $\forall$'s move by placing a pebble in K_n. There are only n pebbles, so $\forall$ will never defeat her. But $\forall$ can win $EF^\omega_\omega(A, B)$, by placing successive pebbles on distinct elements of A. At the outset, $\exists$ must choose which K_n to respond in; and from then on in the game she must stick with this same K_n. However large n is, it will be finite; after n rounds, she will have pebbled every element of this copy, and she will lose in the next round.

Thus, by theorem 39 and proposition 31, we have $\mathcal{A}_{A,B} \in \bigcap_{n<\omega} \mathsf{B}_n \setminus \mathsf{B}_\omega$.

We have $\bigcap_{n<\omega} \mathsf{B}_n = \bigcap_{n<\omega} \mathsf{CRA}_n$ by theorem 22. $\Box$

7 Conclusion

We have analysed the classes RA_n of Maddux, shown that there is a representation theory for them, and seen that for $n \geq 5$, RA_n is not finitely based over RA_{n-1}, nor is B_n elementary.

The following problems arise. In this paper we have only been interested in finite or countable dimensional relational bases, but there is no reason to stop there:

PROBLEM 1 *Can theorem 50 (and other results such as proposition 18) be generalised to* $\mathsf{B}_\kappa, \mathsf{CRA}_\kappa$ *for uncountable* κ*?*

It is worth mentioning that for $n \geq 6$, RA_n is not closed under completions (and hence, by a result of Venema [Ven97], not Sahlqvist-axiomatisable). This can be shown by a slight adaptation of the example used to prove the same for RRA in [Hod97]. RA_4 (RA) is closed under completions, by a result of Monk [Mon70]; so is RA_3 (SA), as Maddux showed [Mad91c, theorem 36].

PROBLEM 2 *Is* RA_5 *closed under completions?*

Questions related to the finite base property are also raised. For $n \geq 3$, the equational theory of the variety RA_n is undecidable: see for example, [AKN+96, theorem 3.1(i)], and, for RA_3 ($= \mathsf{SA}$), [Mad94]. Now we gave a recursive axiomatisation of RA_n in theorem 35. So the validities of RA_n are recursively enumerable. Hence, it cannot be that every equation that is not valid in RA_n fails in some finite algebra in RA_n. For if this were so, then as it is decidable whether a finite algebra is in RA_n (e.g., verify that the equations defining NA or SA hold, then guess an n-dimensional relational basis and check that it is one), the non-valid equations of RA_n would also be recursively enumerable. So the valid ones would form a recursive set, a contradiction.

PROBLEM 3 *For each finite $n \geq 3$, find an equation that is not valid in RA_n but is valid in every finite algebra in RA_n.*

The following question is not answered by the above argument.

PROBLEM 4 *For which $n \geq 3$ is it true that every finite algebra $\mathcal{A} \in \mathsf{RA}_n$ has a finite n-square relativised representation — i.e., $T^n_{\mathcal{A}}$ has a finite model?*

Any RA_n for which it is true is said to have the 'finite algebra on finite base property'. $\mathsf{RA}_\omega = \mathsf{RRA}$ certainly does not have the property: for example, the 'point algebra', used in the artificial planning community and described for example in [HH97d], has eight elements but does not have any finite ω-square relativised representation. We remark that with respect to a slightly different notion of n-square relativised representation, the answer to the problem is positive for all n; this can be shown by Herwig's theorem [Her95, Her98b], using techniques of [AHN99].

Here is a vague question. Consider the following evidence:

- CRA_3 and CRA_4 are elementary, consisting of the atomic algebras in RA_n; a similar result holds for boolean algebras.

  But for $n \geq 5$, CRA_n is not elementary; a similar result holds for RRA.

- RA_3 and RA_4 are finitely axiomatisable, as are boolean algebras.

  But for $n \geq 5$, RA_n is not finitely axiomatisable; a similar result holds for RRA.

- RA_3 and RA_4 are closed under completions, as are boolean algebras.

  But for $n \geq 6$, RA_n is not closed under completions, and neither is RRA.

- On the other hand, the question '$\mathcal{A} \in \mathsf{RA}_n$' for finite $\mathcal{A}$ is decidable if n is finite, and undecidable if $n = \omega$.

Broadly, for $n = 3$ it seems that the algebras in RA_n and B_n behave like boolean algebras, while for $n \geq 5$, or maybe 6, they become more like full-blooded representable relation algebras.

PROBLEM 5 *Find other evidence bearing on this. Is it really true? Why?*

It would be worth investigating the following analogue of Maddux's important question [Mad82, problem 2.3] for RRA; answering it for RA_n may prove to be easier.

PROBLEM 6 *For $n \geq 5$, let $Str\mathsf{RA}_n$ denote the class of all atom structures $\mathcal{C}$ such that the complex algebra $Cm\mathcal{C}$ over $\mathcal{C}$ is in RA_n. For which (finite) n is $Str\mathsf{RA}_n$ an elementary class?*

$Str\mathsf{RA}_3$ and $Str\mathsf{RA}_4$ are elementary classes and explicit axioms are known [Mad82, theorem 2.2(4,5)].

In a similar vein:

PROBLEM 7 *For which (finite) n does RA_n have a canonical equational axiomatisation?*

This was asked for RRA by Venema (and perhaps others). A canonical axiomatisation is one by canonical equations; an equation ε is canonical if $\mathcal{A} \models \varepsilon \Rightarrow \mathcal{A}^+ \models \varepsilon$. Of course, RA_3 and RA_4 are Sahlqvist-axiomatised, and Sahlqvist equations are canonical. But as we said above, for $n \geq 6$, RA_n has no Sahlqvist axiomatisation.

References

[AHN99] H Andréka, I Hodkinson, and I Németi. Finite algebras of relations are representable on finite sets. *J. Symbolic Logic*, 64:243–267, 1999.

[AKN+96] H Andréka, A Kurucz, I Németi, I Sain, and A Simon. Causes and remedies for undecidability in arrow logics and in multi-modal logics. In M. Marx, L. Pólos, and M. Masuch, editors, *Arrow Logic and Multi-Modal Logic*, pages 63–99. CSLI Publications and FoLLI, 1996.

[Bar77] J Barwise. On Moschovakis closure ordinals. *Journal of Symbolic Logic*, 42:292–296, 1977.

[GS53] D Gale and F Stewart. Infinite games with perfect information. *Contributions to the theory of games II, Annals of mathematical studies*, 28:291–296, 1953.

[Her95] B. Herwig. Extending partial isomorphisms on finite structures. *Combinatorica*, 15:365–371, 1995.

[Her98b] B. Herwig. Extending partial isomorphisms for the small index property of many ω–categorical structures. *Israel J. Math.*, 107:93–124, 1998.

[HH97a] R Hirsch and I Hodkinson. Axiomatising various classes of relation and cylindric algebras. *Logic Journal of the IGPL*, 5(2):209–229, 1997.

[HH97b] R Hirsch and I Hodkinson. Complete representations in algebraic logic. *Journal of Symbolic Logic*, 62:816–847, 1997.

[HH97d] R Hirsch and I Hodkinson. Step by step — building representations in algebraic logic. *Journal of Symbolic Logic*, 62:225–279, 1997.

[HH98a] R Hirsch and I Hodkinson. Relation algebras from cylindric algebras, I. Preprint, 1998. Available at `http://theory.doc.ic.ac.uk:80/tfm/papers/HodkinsonIM/public_html/papers.html`

[HH98b] R Hirsch and I Hodkinson. Relation algebras from cylindric algebras, II. Preprint, 1998. Available at `http://www.cs.ucl.ac.uk/staff/R.Hirsch/publications.html`

[HH99c] R Hirsch and I Hodkinson. Representability is not decidable for finite relation algebras. *Trans. Amer. Math. Soc.*, 1999? To appear.

[Hir95] R Hirsch. Completely representable relation algebras. *Bulletin of the interest group in propositional and predicate logics*, 3(1):77–92, 1995.

[HMT85] L Henkin, J D Monk, and A Tarski. *Cylindric Algebras Part II.* North-Holland, 1985.

[Hod93] W Hodges. *Model theory.* Vol. 42 of: Encyclopedia of mathematics and its applications. Cambridge University Press, 1993.

[Hod97] I Hodkinson. Atom structures of cylindric algebras and relation algebras. *Annals of Pure and Applied Logic*, 89:117–148, 1997.

[Imm82] N Immerman. Upper and lower bounds for first-order expressibility. *J. Comput. System Sci.*, 25:76–98, 1982.

[Jón91] B Jónsson. The theory of binary relations. In H Andréka, J D Monk, and I Németi, editors, *Algebraic Logic*, pages 245–292. North-Holland, 1991.

[JT48] B Jónsson and A Tarski. Representation problems for relation algebras. *Bulletin of the American Mathematical Society*, 54:80,1192, 1948. Abstract 89.

[JT51] B Jónsson and A Tarski. Boolean algebras with operators, I. *American Journal of Mathematics*, 73:891–939, 1951.

[JT52] B Jónsson and A Tarski. Boolean algebras with operators, II. *American Journal of Mathematics*, 74:127–162, 1952.

[Lyn50] R Lyndon. The representation of relational algebras. *Annals of Mathematics*, 51(3):707–729, 1950.

[Lyn56] R Lyndon. The representation of relation algebras, II. *Annals of Mathematics*, 63(2):294–307, 1956.

[Mad78] R D Maddux. *Topics in Relation Algebra.* PhD thesis, University of California, Berkeley, 1978.

[Mad82] R D Maddux. Some varieties containing relation algebras. *Transactions of the AMS*, 272(2):501–526, 1982.

[Mad83] R D Maddux. A sequent calculus for relation algebras. *Annals of Pure and Applied Logic*, 25:73–101, 1983.

[Mad89] R D Maddux. Nonfinite axiomatizability results for cylindric and relation algebras. *Journal of Symbolic Logic*, 54(3):951–974, 1989.

[Mad91a] R D Maddux. Introductory course on relation algebras, finite-dimensional cylindric algebras, and their interconnections. In H Andréka, J D Monk, and I Németi, editors, *Algebraic Logic*, pages 361–392. North-Holland, 1991.

[Mad91b] R D Maddux. The origin of relation algebras in the development and axiomatization of the calculus of relations. *Studia Logica*, 3/4:421–456, 1991.

[Mad91c] R D Maddux. Pair-dense relation algebras. *Trans. Amer. Math. Soc.*, 328(1):83–131, 1991.

[Mad92] R Maddux. Relation algebras of every dimension. *Journal of Symbolic Logic*, 57:1213–1229, 1992.

[Mad94] R Maddux. Undecidable semi-associative relation algebras. *Journal of Symbolic Logic*, 59:398–418, 1994.

[McK66] R McKenzie. *The representation of relation algebras.* PhD thesis, University of Colorado at Boulder, 1966.

[Mon64] J D Monk. On representable relation algebras. *Michigan Mathematics Journal*, 11:207–210, 1964.

[Mon70] J D Monk. Completions of boolean algebras with operators. *Mathematische Nachrichten*, 46:47–55, 1970.

[Ném86] I Németi. Free algebras and decidability in algebraic logic. Doctoral Dissertation with the Academy, Budapest, 1986.

[Ném95] I Németi. Decidable versions of first-order logic. In L. Csirmaz, D. Gabbay, and M. de Rijke, editors, *Logic Colloquium '92*, pages 177–241. CSLI Publications & FoLLI, 1995.

[Poi82] B Poizat. Deux ou trois choses que je sais de L_n. *Journal of Symbolic Logic*, 47:641–658, 1982.

[SV97] V Stebletsova and Y Venema. Q-algebras. *Algebra Universalis,* 40:19–49, 1998.

[Ven97] Y Venema. Atom structures and Sahlqvist equations. *Algebra Universalis*, 38:185–199, 1997.

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